Finite Field-Dependent BRST-antiBRST Transformations: Jacobians and Application to the Standard Model

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Abstract

We continue our research Nucl.Phys B888, 92 (2014); Int. J. Mod. Phys. A29, 1450159 (2014); Phys. Lett. B739, 110 (2014); Int. J. Mod. Phys. A30, 1550021 (2015) and extend the class of finite BRST-antiBRST transformations with odd-valued parameters λ_a, $a = 1, 2$, introduced in these works. In doing so, we evaluate the Jacobians induced by finite BRST-antiBRST transformations linear in functionally-dependent parameters, as well as those induced by finite BRST-antiBRST transformations with arbitrary functional parameters. The calculations cover the cases of gauge theories with a closed algebra, dynamical systems with first-class constraints, and general gauge theories. The resulting Jacobians in the case of linearized transformations are different from those in the case of polynomial dependence on the parameters. Finite BRST-antiBRST transformations with arbitrary parameters induce an extra contribution to the quantum action, which cannot be absorbed into a change of the gauge. These transformations include an extended case of functionally-dependent parameters that implies a modified compensation equation, which admits non-trivial solutions leading to a Jacobian equal to unity. Finite BRST-antiBRST transformations with functionally-dependent parameters are applied to the Standard Model, and an explicit form of functionally-dependent parameters λ_a is obtained, providing the equivalence of path integrals in any 3-parameter $R_{\boldsymbol{\xi}}$-like gauges. The Gribov–Zwanziger theory is extended to the case of the Standard Model, and a form of the Gribov horizon functional is suggested in the Landau gauge, as well as in $R_{\boldsymbol{\xi}}$-like gauges, in a gauge-independent way using field-dependent BRST-antiBRST transformations, and in $R_{\boldsymbol{\xi}}$-like gauges using transverse-like non-Abelian gauge fields.

Keywords: Yang–Mills theory, general gauge theory, BRST-antiBRST quantization, constrained dynamical systems, field-dependent BRST-antiBRST transformations, Standard Model, Gribov ambiguity

1 Introduction

Recently, in the articles [1, 2, 3, 4], we have proposed an extension of BRST-antiBRST transformations [5, 6, 7, 8] to the case of finite (both global and field-dependent) parameters for Yang–Mills and general gauge theories in the framework of the generalized Hamiltonian [9, 10] – see also [11] – and Lagrangian [12, 13, 14] BRST-antiBRST quantization schemes. The idea of "finiteness" incorporates into finite transformations a new term being quadratic in the transformation parameters λ_a, thereby lifting BRST-antiBRST transformations from the algebraic level to the

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group level, which has been discussed also in [15, 16]. BRST transformations [17, 18, 19] in both the Lagrangian [20, 21] and generalized Hamiltonian [19, 22, 23] quantization schemes – described by a single odd-valued parameter μ and trivially lifted from the algebraic form $\delta\phi^A = \phi^A \overleftarrow{s} \mu$ to the finite (group) form $\Delta\phi^A = \phi^A[1 - \exp(\overleftarrow{s}\mu)]$, with $\overleftarrow{s}^2 = 0$, in view of the nilpotency property $\mu^2 = 0$ – have first been suggested in Yang–Mills theories for field-dependent parameters in [24, 25]; see also [26, 27]. The introduction of such transformations is based on a functional equation for the infinitesimal parameter, providing the invariance of the integrand of the vacuum functional (in the path integral representation based on the Faddeev–Popov rules [28]) under a change of variables $\phi^A \to \phi'^A\,(\phi; \mu(\phi))$ which preserves the quantum action in gauges described by different gauge Fermions $\psi(\phi)$ and $(\psi + \Delta\psi)(\phi)$, related by the given change. The problem of finding a relation between the different forms of the Faddeev–Popov quantum action in different gauges, expressed by an exact solution $\mu = \mu(\phi; \Delta\psi)$ to a functional equation for a finite field-dependent odd-valued parameter (which ensures the preservation of the integrand) has been solved for the Yang–Mills theory in the article [29]. The respective problem for constrained dynamical systems has been solved in [30], and for general gauge theories, in [31, 32], on the basis of finding the Jacobian of a change of variables induced by the respective field-dependent BRST transformations.

As we return to the approach of [1, 2, 3, 4] – reviewed and extended in [33, 34] – we notice, in the first place, that it allows one to realize the complete BRST-antiBRST invariance of the integrand in the vacuum functional. The functionally-dependent parameters $\lambda_a = s_a\Lambda$, induced by an even-valued functional Λ and by an Sp(2)-doublet of BRST-antiBRST generators s_a, provide an explicit correspondence (due to the compensation equation for the corresponding Jacobian) between a choice of Λ and a transition from the vacuum functional of a given theory in a certain gauge induced by a gauge Boson F_0 to the same theory in a different gauge induced by another gauge Boson F. This becomes a key instrument of a BRST-antiBRST approach that allows one to consistently examine the notion of "soft BRST-antiBRST symmetry breaking" [4], extending the concept of "soft BRST symmetry breaking" [35, 36, 37] in the framework of Lagrangian BRST quantization [21], which implies an extension of the quantum action given by the Lagrangian BRST-antiBRST recipe [12, 13, 14] by a BRST-antiBRST non-invariant term, which is then employed in the concept of effective average action in the *functional renormalization group* approach [38, 39, 40, 41] in [31, 42], as well as the interacting Fermi systems [43] and in the elimination of residual gauge invariance in the deep IR region, known as Gribov copies [44]. Finite field-dependent BRST and BRST-antiBRST transformations, respectively, in soft BRST and BRST-antiBRST symmetry breaking allow one to solve the consistency problem for the Lagrangian quantization methods from the viewpoint of gauge-independence for the conventional S-matrix in non-Abelian gauge theories, namely, in determining a BRST(-antiBRST) non-invariant addition to the corresponding quantum action – known as the Gribov horizon functional [44] which is initially given by the Landau gauge in the Gribov–Zwanziger theory [45, 46] – by using any other gauge, including the one-parameter R_ξ-gauges in the BRST [31, 47, 48] and BRST-antiBRST [1] settings.

In the case of finite BRST-antiBRST transformations, the functionally-dependent parameters λ_a chosen as solutions to the compensation equation, relating the Jacobian to a finite change of the gauge condition, turn out to establish a coincidence of vacuum functionals also in first-class constraint dynamical systems in different gauges. This has been shown explicitly in the case of Yang–Mills theories, thereby providing the unitarity of the conventional S-matrix in Lagrangian formalism within different gauges [2]. At the same time, we have examined [3] the Freedman–Townsend model [49], being the case of a first-stage reducible gauge theory (of a non-Abelian antisymmetric tensor field), in the path integral representation, starting from a reference frame with a certain gauge Boson F_0, and reaching the same integrand, by using finite field-dependent BRST-antiBRST transformations, in a different reference frame with another gauge Boson F, depending on 3 gauge parameters.

It should be noted that we have so far examined the finite field-dependent BRST-antiBRST transformations with

functionally-dependent parameters of the form $\lambda_a = s_a\Lambda$. However, in the conclusions of [1, 4] we have announced that an interesting problem, left outside the scope of [1, 4], is the evaluation of Jacobians for finite field-dependent BRST-antiBRST transformations with a functionally-independent Sp(2)-doublet of arbitrary odd-valued parameters λ_a, i.e., not being induced by any even-valued functional Λ, $\lambda_a \not\equiv s_a\Lambda$. Such Jacobians have not been found explicitly in [15, 16] by using solutions of the equations involved. Another interesting task is the evaluation of Jacobians of linearized transformations, i.e., those without the term being quadratic in the parameters λ_a. It then appears to be important to apply the results involving the study of finite BRST(-antiBRST) field-dependent transformations to the realistic physical model being an example of the Yang–Mills theory interacting with scalar and spinor matter fields and known as the Standard Model [50, 51, 52] – see also [53, 54, 55, 56] – which describes the known spectrum of the elementary particles corresponding to the three fundamental interactions: electromagnetic, weak and strong, whose cornerstone, the Higgs Boson [57, 58, 59, 60], has been discovered [61, 62] at the LHC in July 2012, with the present estimation [63] of its mass being $m_{\mathrm{H}} = (125,09 \pm 0,24)$GeV.

Based on the above reasons, we examine the following problems related to gauge theories in the Lagrangian and generalized Hamiltonian descriptions:

1. evaluation of the Jacobian for a change of variables in the vacuum functional corresponding to *linearized finite field-dependent BRST-antiBRST transformations* in Yang–Mills theories and first-class constraint dynamical systems;

2. evaluation of the Jacobian for a change of variables in the vacuum functional corresponding to *finite field-dependent BRST-antiBRST transformations* with arbitrary functional parameters, $\lambda_a(\phi)$, $s^2\lambda_a(\phi) \not\equiv 0$, in Yang–Mills theories, first-class constraint dynamical systems, and general gauge theories, and investigation of its influence on the structure of the quantum action;

3. construction of the parameters λ_a of finite field-dependent BRST-antiBRST transformations in the Lagrangian action of the Standard Model, which generates a change of the gauge in the path integral within a class of linear 3-parameter $R_{\boldsymbol{\xi}}$-like gauges, realized in terms of an even-valued gauge functionals $F_{\boldsymbol{\xi}}$, with $(\xi_1, \xi_2, \xi_3) = \mathbf{0}, \mathbf{1}$, corresponding to the Landau and Feynman (covariant) gauges, respectively;

4. construction of the Gribov–Zwanziger theory for the BRST-antiBRST Lagrangian formulation of the Standard Model, including the horizon functional $h_{\boldsymbol{\xi}}$ in arbitrary $R_{\boldsymbol{\xi}}$-like gauges by means of finite field-dependent BRST-antiBRST transformations, starting from the BRST-antiBRST non-invariant functional $h_{\mathbf{0}}$ given in the Landau gauge and realized in terms of the even-valued functional $F_{\mathbf{0}}$.

5. construction of an horizon functional $h^T_{\boldsymbol{\xi}}$ for the Standard Model in the Gribov–Zwanziger theory with arbitrary $R_{\boldsymbol{\xi}}$-like gauges by means of a Hermitian extension of the corresponding Faddeev–Popov operator (or, equivalently, in terms of transverse-like components of non-Abelian gauge fields), following the recipe of [64] for a Yang–Mills theory with an $SU(N)$ gauge group.

The work is organized as follows. In Section 2, we give an overview of the ingredients of finite field-dependent BRST-antiBRST transformations [1, 2, 3, 4] in theories with a closed gauge algebra, as well as in first-class constraint dynamical systems and general gauge theories. In Section 3, we consider an evaluation of the Jacobian for a change of variables in the vacuum functional given by finite field-dependent BRST-antiBRST transformations being *linear* in functionally-dependent parameters $\lambda_a = s_a\Lambda$ for Yang–Mills theories and first-class constraint dynamical systems in generalized Hamiltonian formalism. In Section 4, we consider an evaluation of the Jacobian for a change of variables in the vacuum functional given by finite field-dependent BRST-antiBRST transformations with arbitrary

parameters λ_a. In Section 5, we consider an application of finite BRST-antiBRST transformations to the Standard Model. In Appendices A and B, we examine the respective details of calculations for linearized finite BRST-antiBRST transformations with functionally-dependent parameters, as well as for finite BRST-antiBRST transformations with arbitrary parameters. In Discussion, we suggest another form of the Gribov horizon functional in the covariant gauge and make concluding remarks. As a rule, we use the conventions of our previous works [1, 2, 3, 4] and the generally accepted definition [65] of functional integrals for quasi-Gaussian functionals, which is well justified in perturbation theory, see, e.g., [66]. Notice that Sections 3, 4 do not need to use the operation of functional integration in itself, but only the definition of a functional Jacobian. Unless otherwise specified, derivatives with respect to the fields are taken from the right, and those with respect to the corresponding sources and antifields are taken from the left. Left-handed derivatives with respect to the fields are labelled by the subscript "l", whereas right-handed derivatives with respect to the antifields are labelled by the subscript "r". Derivatives with respect to the phase-space variables and the variables of the triplectic manifold are understood as taken from the right. Depending on the convenience, we use two forms of notation for the BRST-antiBRST generators: s_a and $\overleftarrow{s}_a$, which are related by $s_a A \equiv A\overleftarrow{s}_a$, where A is an arbitrary functional. The raising and lowering of $\mathrm{Sp}\,(2)$ indices, $s^a = \varepsilon^{ab}s_b$, $s_a = \varepsilon_{ab}s^b$, is carried out with the help of a constant antisymmetric tensor ε^{ab}, $\varepsilon^{ac}\varepsilon_{cb} = \delta^a_b$, subject to the normalization condition $\varepsilon^{12} = 1$. The Grassmann parity of any homogeneous quantity B is denoted by $\varepsilon(B)$.

2 Finite BRST-antiBRST Transformations

In this section, we examine the case of finite BRST-antiBRST transformations realized in different spaces of quantum field theory: the configuration space of Yang–Mills theories, the phase space of arbitrary dynamical systems with first-class constraints, and the triplectic space of general gauge theories in Lagrangian formalism.

2.1 Yang–Mills Theories in Lagrangian Formalism

The generating functional of Green's functions corresponding to irreducible gauge theories with a closed algebra in BRST-antiBRST Lagrangian quantization [12, 13] is given by

$$Z(J) = \int d\phi\ \exp\left\{\frac{i}{\hbar}\left[S_F\,(\phi) + J_A\phi^A\right]\right\} \equiv \int \mathcal{I}^F_\phi \exp\left(\frac{i}{\hbar}J_A\phi^A\right)\ . \tag{2.1}$$

Here, $\hbar$ is the Planck constant, whereas the quantum action $S_F\,(\phi)$,

$$S_F\,(\phi) = S_0\,(A) + F_{,A}Y^A - (1/2)\,\varepsilon_{ab}X^{Aa}F_{,AB}X^{Bb}\ , \tag{2.2}$$

the classical action $S_0\,(A)$, the (admissible) even-valued gauge-fixing functional $F\,(\phi)$, and the functions[1] $X^{Aa}\,(\phi)$, $Y^A\,(\phi)$ are defined in the configuration space $\mathcal{M}_\phi = \{\phi^A\} = \{A^i, C^{\alpha a}, B^\alpha\}$, parameterized by the initial classical fields A^i, $i = 1, \ldots, n$, the Nakanishi–Lautrup fields B^α, $\alpha = 1, \ldots, m < n$, and the ghost-antighost fields $C^{\alpha a}$, organized in $\mathrm{Sp}\,(2)$-doublets with the identification $C^{\alpha a} = (C^{\alpha 1}, C^{\alpha 2}) \equiv (C^\alpha, \bar{C}^\alpha)$. The Grassmann parity is given by

$$\varepsilon(\phi^A) = \varepsilon(A^i, B^\alpha, C^{\alpha a}) = (\varepsilon_i, \varepsilon_\alpha, \varepsilon_\alpha + 1) = \varepsilon_A\ . \tag{2.3}$$

The classical action $S_0\,(A)$ is invariant with respect to the infinitesimal gauge transformations

$$\delta A^i = R^i_\alpha(A)\xi^\alpha \Longrightarrow S_{0,i}(A)R^i_\alpha(A) = 0\ , \tag{2.4}$$

[1] By functions we understand those of the space-time coordinates.

with $R^i_\alpha(A)$ being the generators of the gauge transformations, $\varepsilon(R^i_\alpha) = \varepsilon_i + \varepsilon_\alpha$, and ξ^α being arbitrary functions. The generators $R^i_\alpha(A)$ form a closed gauge algebra, with structure constants $F^\gamma_{\alpha\beta}(A) = \text{const}$ and vanishing quantities $M^{ij}_{\alpha\beta}(A)$ in the general relations [21]

$$
\begin{aligned}
&R^i_{\alpha,j}(A)R^j_\beta(A) - (-1)^{\varepsilon_\alpha\varepsilon_\beta} R^i_{\beta,j}(A)R^j_\alpha(A) = -R^i_\gamma(A)F^\gamma_{\alpha\beta}(A) - S_{0,j}(A)M^{ij}_{\alpha\beta}(A) \ , \\
&F^\gamma_{\alpha\beta} = -(-1)^{\varepsilon_\alpha\varepsilon_\beta} F^\gamma_{\beta\alpha} \ , \quad M^{ij}_{\alpha\beta} = -(-1)^{\varepsilon_i\varepsilon_j} M^{ji}_{\alpha\beta} = -(-1)^{\varepsilon_\alpha\varepsilon_\beta} M^{ij}_{\beta\alpha} \, .
\end{aligned}
\tag{2.5}
$$

In a first-rank gauge theory with a closed algebra, the functions $X^{Aa}(\phi)$, $Y^A(\phi)$ in (2.2) are given by [12]

$$
X^{Aa} = \left(X^{ia}_1, X^{\alpha a}_2, X^{\alpha ab}_3\right) \ , \qquad Y^A = \left(Y^i_1, Y^\alpha_2, Y^{\alpha a}_3\right) \ , \tag{2.6}
$$

$$
\begin{aligned}
&X^{ia}_1 = R^i_\alpha C^{\alpha a} \ , && X^{\alpha a}_2 = -\frac{1}{2}F^\alpha_{\gamma\beta}B^\beta C^{\gamma a} - \frac{1}{12}(-1)^{\varepsilon_\beta}\left(2F^\alpha_{\gamma\beta,j}R^j_\rho + F^\alpha_{\gamma\sigma}F^\sigma_{\beta\rho}\right)C^{\rho b}C^{\beta a}C^{\gamma c}\varepsilon_{cb} \ , \\
&X^{\alpha ab}_3 = -\varepsilon^{ab}B^\alpha - \frac{1}{2}(-1)^{\varepsilon_\beta} F^\alpha_{\beta\gamma}C^{\gamma b}C^{\beta a} \ , && Y^i_1 = R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha} R^i_{\alpha,j}R^j_\beta C^{\beta b}C^{\alpha a}\varepsilon_{ab} \ , \\
&Y^\alpha_2 = 0 \ , && Y^{\alpha a}_3 = -2X^{\alpha a}_3
\end{aligned}
\tag{2.7}
$$

and in Yang–Mills theories they assume the following representation:

$$
\begin{aligned}
&X^{\mu ma}_1 = D^{\mu mn}C^{na} \ , && Y^{\mu m}_1 = D^{\mu mn}B^n + \frac{1}{2}f^{mnl}C^{la}D^{\mu nk}C^{kb}\varepsilon_{ba} \ , \\
&X^{ma}_2 = -\frac{1}{2}f^{mnl}B^lC^{na} - \frac{1}{12}f^{mnl}f^{lrs}C^{sb}C^{ra}C^{nc}\varepsilon_{cb} \ , && Y^m_2 = 0 \ , \\
&X^{mab}_3 = -\varepsilon^{ab}B^m - \frac{1}{2}f^{mnl}C^{lb}C^{na} \ , && Y^{ma}_3 = f^{mnl}B^lC^{na} + \frac{1}{6}f^{mnl}f^{lrs}C^{sb}C^{ra}C^{nc}\varepsilon_{cb} \ ,
\end{aligned}
\tag{2.8}
$$

corresponding to the generators R^i_α and structure functions $F^\gamma_{\alpha\beta}$,

$$
R^{mn}_\mu(x;y) = D^{mn}_\mu(x)\delta(x-y)\,, \quad D^{mn}_\mu = \delta^{mn}\partial_\mu + f^{mln}A^l_\mu\,, \quad F^\gamma_{\alpha\beta} = f^{lmn}\delta(x-z)\delta(y-z)\,, \tag{2.9}
$$

written down in terms of a covariant derivative D^{mn}_μ and completely antisymmetric structure constants f^{lmn} related to a compact subalgebra of an $su(N)$ Lie algebra.

The quantum action S_F, the integration measure $d\phi$, and thereby also the integrand $\mathcal{I}^F_\phi$, are invariant under BRST-antiBRST transformations, which are infinitesimal transformations with an Sp(2)-doublet of constant odd-valued parameters μ_a,

$$
\delta\phi^A = \left(s^a\phi^A\right)\mu_a \ , \quad s^a\phi^A = X^{Aa} \ , \tag{2.10}
$$

where s^a are the generators of BRST-antiBRST transformations. Starting from this point, the invariance of the integrand $\mathcal{I}^F_\phi$ in the case of finite constant values of the corresponding anticommuting parameters λ_a is achieved by solving the equation $G(\phi+\Delta\phi) = G(\phi)$ for an arbitrary regular functional $G(\phi)$ subject to BRST-antiBRST invariance, $s^aG = 0$. This solution has the form of a finite (polynomial in λ_a) BRST-antiBRST transformation [1]

$$
\Delta\phi^A = \phi^A + X^{Aa}\lambda_a - \frac{1}{2}Y^A\lambda^2 = \left(s^a\phi^A\right)\lambda_a + \frac{1}{4}\left(s^2\phi^A\right)\lambda^2 \tag{2.11}
$$

and implies that a finite variation $\Delta\phi^A$ includes the generators of BRST-antiBRST transformations $\left(s^1, s^2\right)$, as well as their commutator $s^2 = \varepsilon_{ab}s^bs^a = s^1s^2 - s^2s^1$, being the generator of mixed BRST-antiBRST transformations. Equivalently, (2.11) can be represented as a group transformation in the configuration space $\mathcal{M}_\phi$,

$$
\phi^A \to \phi'^A = \phi^A\left(1 + \overleftarrow{s}^a\lambda_a + \frac{1}{4}\overleftarrow{s}^2\lambda^2\right) = \phi^A\exp\left(\overleftarrow{s}^a\lambda_a\right) \ , \quad s^a\phi^A = \phi^A\overleftarrow{s}^a, \quad \overleftarrow{s}^2 \equiv \overleftarrow{s}^a\overleftarrow{s}_a = s^2 \equiv s_as^a \ , \tag{2.12}
$$

where the set of elements $\{g(\lambda)\} = \{\exp(\overleftarrow{s}^a\lambda_a)\}$ forms an Abelian two-parameter supergroup with odd-valued generating elements λ_a. This circumstance can also be justified by the Frobenius theorem, which deals with an implementation of anticommuting generators $\overleftarrow{s}^a$ in terms of vector fields $\overleftarrow{s}^a(\Gamma) = \frac{\overleftarrow{\delta}}{\delta\Gamma^p}(\Gamma^p\overleftarrow{s}^a)$ in a certain configuration space $\mathcal{M}_\Gamma$. The BRST-antiBRST invariance of $\mathcal{I}^F_\phi$ implies the relation

$$\mathcal{I}^F_{\phi g(\lambda)} = \mathcal{I}^F_\phi. \tag{2.13}$$

which can be established by the fact [1] that the global finite transformations (corresponding to $\lambda_a = \text{const}$) respect the integration measure:

$$\mathrm{Sdet}\left(\frac{\delta\phi'}{\delta\phi}\right) = 1 \Longrightarrow d\phi' = d\phi\ . \tag{2.14}$$

For finite field-dependent transformations, it has been established [1] that in the particular case of functionally-dependent parameters $\lambda_a = \Lambda\overleftarrow{s}_a$, $s^1\lambda_1 + s^2\lambda_2 = -s^2\Lambda$, with a certain even-valued potential, $\Lambda = \Lambda(\phi)$, whose introduction has been inspired by infinitesimal field-dependent BRST-antiBRST transformations induced by the parameters [12]

$$\mu_a = \frac{i}{2\hbar}\varepsilon_{ab}\left(\Delta F\right)_{,A} X^{Ab} = \frac{i}{2\hbar}\left(s_a\Delta F\right)\ , \tag{2.15}$$

the vacuum functional $Z_F(0)$ is gauge-independent: $Z_{F+\Delta F}(0) = Z_F(0)$. Namely, in the case of finite field-dependent transformations with group-like elements $g(\Lambda\overleftarrow{s}_a)$ whose set forms a nonlinear non-Abelian group-like structure[2] the superdeterminant of a change of variables is given by

$$\mathrm{Sdet}\left[\frac{\delta(\phi g(\Lambda\overleftarrow{s}_a)}{\delta\phi}\right] = \exp\left[\Im(\phi)\right]\ , \quad \text{where} \quad \Im(\phi) = -2\ln\left[1 - \frac{1}{2}s^2\Lambda(\phi)\right]\ , \tag{2.16}$$

$$d\phi' = d\phi\ \exp\left[\frac{i}{\hbar}\left(-i\hbar\Im\right)\right] = d\phi\ \exp\left\{\frac{i}{\hbar}\left[i\hbar\ln\left(1 - \frac{1}{2}s^2\Lambda\right)^2\right]\right\}\ . \tag{2.17}$$

The invariance of the quantum action $S_F(\phi)$ with respect to (2.11) implies that the change $\phi^A \to \phi'^A = \phi^A g(\lambda(\phi))$ induces in (2.1) the following transformation of the integrand $\mathcal{I}^F_\phi$:

$$\mathcal{I}^F_{\phi g(\lambda(\phi))} = d\phi\ \exp\left[\Im(\phi)\right]\exp\left[(i/\hbar)\, S_F\left(\phi\left(g\lambda(\phi)\right)\right)\right] = d\phi\ \exp\left\{(i/\hbar)\left[S_F(\phi) - i\hbar\Im(\phi)\right]\right\}\ , \tag{2.18}$$

whence

$$\mathcal{I}^F_{\phi g(\lambda(\phi))} = d\phi\ \exp\left\{(i/\hbar)\left[S_F(\phi) + i\hbar\ \ln\left(1 - \Lambda\overleftarrow{s}^2/2\right)^2\right]\right\}\ . \tag{2.19}$$

Next, due to the explicit form of the initial quantum action $S_F = S_0 - (1/2)\, F\overleftarrow{s}^2$, the BRST-antiBRST-exact contribution $i\hbar\ln\left(1 + s^a s_a\Lambda/2\right)^2$ to the quantum action S_F can be interpreted as a change of the gauge-fixing functional made in the original integrand $\mathcal{I}^F_\phi$,

$$i\hbar\ \ln\left(1 + s^a s_a\Lambda/2\right)^2 = s^a s_a\left(\Delta F/2\right) \tag{2.20}$$

$$\Longrightarrow \mathcal{I}^F_{\phi g(\lambda(\phi))} = d\phi\ \exp\left\{(i/\hbar)\left[S_0 + (1/2)\, s^a s_a\left(F + \Delta F\right)\right]\right\} = \mathcal{I}^{F+\Delta F}_\phi, \tag{2.21}$$

with a certain $\Delta F(\phi|\Lambda)$, whose correspondence to $\Lambda(\phi)$ is established by the relation (2.20), which is also known as the compensation equation for an unknown parameter $\Lambda(\phi)$ and which thereby provides the gauge-independence of

[2]For BRST-antiBRST-closed (in particular, BRST-antiBRST-exact) functional parameters $\lambda_a(\phi) = \Lambda_a(\phi)\overleftarrow{s}^2$ with odd-valued functionals $\Lambda_a(\phi)$, the subset $g(\Lambda_a\overleftarrow{s}^2)$ forms an Abelian subgroup in $g(\Lambda\overleftarrow{s}_a)$ and thereby in $g(\lambda_a(\phi))$. Indeed, the choice $\Lambda = 2s^a\Lambda_a$, in view of $\lambda_b(\phi)\overleftarrow{s}^a = 0$ provided by $\overleftarrow{s}^a\overleftarrow{s}^b\overleftarrow{s}^c \equiv 0$, implies that $g(\Lambda^1_a\overleftarrow{s}^2)g(\Lambda^2_a\overleftarrow{s}^2) = g(\Lambda^2_a\overleftarrow{s}^2)g(\Lambda^1_a\overleftarrow{s}^2)$ and $g(\Lambda^1_a\overleftarrow{s}^2)g(-\Lambda^1_a\overleftarrow{s}^2) = 1$, for any odd-valued functionals Λ^i_a , $i = 1, 2$, with the unit element “1”.

the vacuum functional, $Z_F(0) = Z_{F+\Delta F}(0)$. An explicit solution of (2.20), satisfying the solvability condition due to the BRST-antiBRST-exactness of both sides (up to BRST-antiBRST-exact terms), is given by

$$\Lambda\left(\phi|\Delta F\right) = 2\Delta F\left(s^a s_a \Delta F\right)^{-1}\left[\exp\left(\frac{1}{4i\hbar}s^b s_b \Delta F\right) - 1\right] = \frac{1}{2i\hbar}\Delta F\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}s^a s_a \Delta F\right)^n \ . \tag{2.22}$$

Conversely, having considered the equation (2.20) for an unknown ΔF with a given Λ, we obtain

$$\Delta F\left(\phi\right) = 4i\hbar\ \Lambda\left(\phi\right)\left(s^a s_a \Lambda\left(\phi\right)\right)^{-1}\ln\left(1 + s^a s_a \Lambda\left(\phi\right)/2\right)\ , \tag{2.23}$$

and therefore a field-dependent transformation with the parameters $\lambda_a = s_a\Lambda$,

$$\lambda_a = \frac{1}{2i\hbar}\left(\Delta F \overleftarrow{s}_a\right)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}\Delta F \overleftarrow{s}^2\right)^n\ , \tag{2.24}$$

amounts to a precise change of the gauge-fixing functional.

In view of (2.22), the property (2.21) implies a so-called modified Ward identity [4], depending on field-dependent parameters $\lambda_a = \Lambda\overleftarrow{s}_a$ and thereby also on a finite change of the gauge:

$$\left\langle\left\{1 + \frac{i}{\hbar}J_A\left[X^{Aa}\lambda_a(\Lambda) - \frac{1}{2}Y^A\lambda^2(\Lambda)\right] - \frac{1}{4}\left(\frac{i}{\hbar}\right)^2\varepsilon_{ab}J_A X^{Aa}J_B X^{Bb}\lambda^2(\Lambda)\right\}\left(1 - \frac{1}{2}\Lambda\overleftarrow{s}^2\right)^{-2}\right\rangle_{F,J} = 1\ . \tag{2.25}$$

The property (2.21) also provides a relation which describes the gauge-dependence of $Z_F(J)$ for a finite change $F \to F + \Delta F$:

$$\begin{aligned} Z_{F+\Delta F}(J) - Z_F(J) = Z_F(J)\Big\langle\frac{i}{\hbar}J_A&\left[X^{Aa}\lambda_a\left(\phi| - \Delta F\right) - \frac{1}{2}Y^A\lambda^2\left(\phi| - \Delta F\right)\right] \\ &- (-1)^{\varepsilon_B}\left(\frac{i}{2\hbar}\right)^2 J_B J_A\left(X^{Aa}X^{Bb}\right)\varepsilon_{ab}\lambda^2\left(\phi| - \Delta F\right)\Big\rangle_{F,J}\ . \end{aligned} \tag{2.26}$$

In (2.25), (2.26), the symbol "$\langle\mathcal{A}\rangle_{F,J}$" for a certain functional $\mathcal{A}(\phi)$ denotes a source-dependent average expectation value corresponding to a gauge-fixing functional $F(\phi)$:

$$\langle\mathcal{A}\rangle_{F,J} = Z_F^{-1}(J)\int d\phi\ \mathcal{A}\left(\phi\right)\exp\left\{\frac{i}{\hbar}\left[S_F\left(\phi\right) + J_A\phi^A\right]\right\}\ ,\quad \langle 1\rangle_{F,J} = 1\ . \tag{2.27}$$

In the case of constant λ_a, the relation (2.25) implies an Sp(2)-doublet of the usual Ward identities (at the first order in λ_a) and a derivative identity (at the second order in λ_a), namely,

$$J_A\left\langle X^{Aa}\right\rangle_{F,J} = 0\ ,\quad \left\langle J_A\left[2Y^A + (i/\hbar)\,\varepsilon_{ab}X^{Aa}J_B X^{Bb}\right]\right\rangle_{F,J} = 0\ . \tag{2.28}$$

Below, we intend to study the case of finite field-dependent BRST-antiBRST transformations for Yang–Mills theories in Lagrangian formalism with arbitrary functional parameters, generally assumed to be functionally-independent, $\lambda_a \not\equiv s_a\Lambda$. It is also intended to study the case of finite field-dependent BRST-antiBRST transformations being linear in functionally-dependent parameters of the form $\lambda_a = s_a\Lambda$.

2.2 Dynamical Systems in Generalized Hamiltonian Formalism

The generating functional of Green's functions for dynamical systems with first-class constraints has the form [9, 10]

$$Z_\Phi\left(I\right) = \int d\Gamma\exp\left\{\frac{i}{\hbar}\int dt\left[\frac{1}{2}\Gamma^p(t)\omega_{pq}\dot{\Gamma}^q(t) - H_\Phi(t) + I(t)\Gamma(t)\right]\right\} \equiv \int\mathcal{I}_\Gamma^\Phi\exp\left\{\frac{i}{\hbar}\int dt\, I(t)\Gamma(t)\right\} \tag{2.29}$$

and determines the vacuum functional $Z_\Phi = Z_\Phi(0)$ at the vanishing external sources $I_p(t)$ to the phase-space variables $\Gamma^p(t)$. In (2.29), integration over time is taken over the range $t_{\rm in} \le t \le t_{\rm out}$; the functions of time $\Gamma^p(t) \equiv \Gamma^p_t$ for $t_{\rm in} \le t \le t_{\rm out}$ are trajectories, $\dot{\Gamma}^p(t) \equiv d\Gamma^p(t)/dt$; the quantities $\omega_{pq} = (-1)^{(\varepsilon_p+1)(\varepsilon_q+1)}\omega_{qp}$ compose an even supermatrix inverse to that with the elements ω^{pq}; the unitarizing Hamiltonian $H_\Phi(t) = H_\Phi(\Gamma(t))$ is determined by four t-local functions: an even-valued function $\mathcal{H}(t)$, with $\mathrm{gh}(\mathcal{H}) = 0$, an Sp(2)-doublet of odd-valued functions $\Omega^a(t)$, with $\mathrm{gh}(\Omega^a) = -(-1)^a$, and an even-valued function $\Phi(t)$, with $\mathrm{gh}(\Phi) = 0$, known as the gauge-fixing Boson,

$$H_\Phi(t) = \mathcal{H}(t) + \frac{1}{2}\varepsilon_{ab}\left\{\{\Phi(t),\Omega^a(t)\}_t\,,\Omega^b(t)\right\}_t\ ,$$

where the functions $\mathcal{H}$, Ω^a are defined in the phase space $\mathcal{M}_\Gamma$ parameterized by the canonical coordinates $\Gamma^p = (\eta, \Gamma_{\rm gh})$, $\varepsilon(\Gamma^p) = \varepsilon(I_p) = \varepsilon_p$, and obey the following generating equations in terms of the Poisson superbracket, $\{\Gamma^p, \Gamma^q\} = \omega^{pq} = \mathrm{const}$, related to the even supermatrix ω^{pq} , with $\omega^{pq} = -(-1)^{\varepsilon_p\varepsilon_q}\omega^{qp}$:

$$\left\{\Omega^a\left(t\right),\Omega^b\left(t\right)\right\}_t = 0\ ,\quad \left\{\mathcal{H}\left(t\right),\Omega^b\left(t\right)\right\}_t = 0\ , \tag{2.30}$$

with account taken of the rule $\{A(t), B(t)\}_t = \{A(\Gamma), B(\Gamma)\}\,|_{\Gamma=\Gamma(t)}$ for any A, B. The functions $\mathcal{H}$, Ω^a are subject to the boundary conditions

$$\mathcal{H}|_{\Gamma_{\rm gh}=0} = H_0\left(\eta\right)\ ,\quad \left.\frac{\delta\Omega^a}{\delta C^{\alpha b}}\right|_{\Gamma_{\rm gh}=0} = \delta^a_b T_\alpha\left(\eta\right)\ , \tag{2.31}$$

where the classical Hamiltonian $H_0 = H_0(\eta)$ and the set of first-class constraints $T_\alpha = T_\alpha(\eta)$, $\varepsilon(T_\alpha) = \varepsilon_\alpha$, of a given dynamical system depend on the classical phase-space variables η, with the involution relations

$$\{H_0, T_\alpha\} = T_\gamma V^\gamma_\alpha\ ,\quad \{T_\alpha, T_\beta\} = T_\gamma U^\gamma_{\alpha\beta}\ ,\quad \text{where}\quad U^\gamma_{\alpha\beta} = -(-1)^{\varepsilon_\alpha\varepsilon_\beta}U^\gamma_{\beta\alpha}\ . \tag{2.32}$$

In (2.31), the variables $\Gamma_{\rm gh}$ contain the entire set of auxiliary variables that correspond to the towers [22] of ghost-antighost coordinates C and Lagrangian multipliers π, as well as their respective conjugate momenta $\mathcal{P}$ and λ, whose structure depends on the reducibility or irreducibility of a given dynamical system and is arranged into Sp(2)-symmetric tensors [9, 10].

In virtue of the generating equations (2.30), the integrand with vanishing sources $\mathcal{I}^\Phi_\Gamma$ in (2.29) is invariant under the infinitesimal BRST-antiBRST transformations [9]

$$\Gamma^p \to \check{\Gamma}^p = \Gamma^p + \left(s^a\Gamma^p\right)\mu_a\ , \tag{2.33}$$

which are realized on phase-space trajectories $\Gamma^p(t)$,

$$\Gamma^p(t) \to \check{\Gamma}^p(t) = \Gamma^p(t) + \{\Gamma^p(t), \Omega^a(t)\}_t\,\mu_a = \Gamma^p(t) + \left(s^a\Gamma^p\right)(t)\,\mu_a\ , \tag{2.34}$$

where μ_a form an Sp(2)-doublet of infinitesimal anticommuting constant parameters, and the generators s^a of BRST-antiBRST transformations, $s^a = \{\bullet, \Omega^a\}$, are anticommuting, nilpotent, and obey the Leibniz rule when acting on the product and the Poisson superbracket:

$$s^as^b + s^bs^a = 0\ ,\ s^as^bs^c = 0\ ,\ s^a\left(AB\right) = \left(s^aA\right)B\left(-1\right)^{\varepsilon_B} + A\left(s^aB\right)\ ,\ s^a\left\{A,B\right\} = \left\{s^aA,B\right\}\left(-1\right)^{\varepsilon_B} + \left\{A,s^aB\right\}. \tag{2.35}$$

The first three relations for s^a are also valid in the case of Lagrangian BRST-antiBRST transformations in Yang–Mills theories. Once again, the achievement of BRST-antiBRST invariance of $\mathcal{I}^\Phi_\Gamma$ in (2.29) with finite constant values of the parameters (now denoted by λ_a) leads to finite transformations [2] of the canonical variables Γ^p,

$$\Gamma^p \to \check{\Gamma}^p = \Gamma^p + \Delta\Gamma^p\ ,\quad \text{where}\quad \Delta\Gamma^p = \left(s^a\Gamma^p\right)\lambda_a + \frac{1}{4}\left(s^2\Gamma^p\right)\lambda^2\ , \tag{2.36}$$

with the same interpretation of both the terms in $\Delta\Gamma^p$ as in the comments that follow the relation (2.11) of Subsection 2.1. In particular, the transformations (2.36) may be represented as group transformations, defined this time in the phase space $\mathcal{M}_\Gamma$ and realized on the canonical coordinates:

$$\Gamma^p \to \check{\Gamma}^p = \Gamma^p \left(1 + \overleftarrow{s}^a \lambda_a + \frac{1}{4}\overleftarrow{s}^2 \lambda^2\right) \equiv \Gamma^p \exp\left(\overleftarrow{s}^a \lambda_a\right) , \tag{2.37}$$

where the operators $\overleftarrow{s}^a$ obey the same notation (2.12) that takes place for their Lagrangian counterparts. The set of elements $\{g(\lambda)\} = \{\exp(\overleftarrow{s}^a \lambda_a)\}$ forms an Abelian two-parameter supergroup with odd-valued generating elements λ_a, acting this time in $\mathcal{M}_\Gamma$, instead of the configuration space $\mathcal{M}$. The transformations (2.36) are realized on phase-space trajectories $\Gamma^p(t)$ as follows:

$$\check{\Gamma}^p(t) = \Gamma^p(t) \exp\left(\overleftarrow{s}^a \lambda_a\right) \Longleftrightarrow \Delta\Gamma^p(t) = \Gamma^p(t)\left[\exp\left(\overleftarrow{s}^a \lambda_a\right) - 1\right] = \left(s^a \Gamma^p\right)(t)\, \lambda_a + \frac{1}{4}\left(s^2 \Gamma^p\right)(t)\, \lambda^2 . \tag{2.38}$$

The BRST-antiBRST invariance of $\mathcal{I}_\Gamma^\Phi$ implies the relation

$$\mathcal{I}^\Phi_{\Gamma g(\lambda)} = \mathcal{I}^\Phi_\Gamma , \tag{2.39}$$

in view of the fact that, due to Liuville's theorem, the measure $d\Gamma$ in (2.29) is right-invariant with respect to the action of the Abelian supergroup, which plays the role of finite canonical transformations, $d\check{\Gamma} = d\Gamma$, and the fact that the Hamiltonian action $S_H(\Gamma) = \int dt \left[\frac{1}{2}\Gamma^p(t)\omega_{pq}\dot{\Gamma}^q(t) - H_\Phi(t)\right]$ is also invariant, $S_H(\Gamma) = S_H(\check{\Gamma})$.

The finite field-dependent transformations (2.38) with parameters $\lambda_a = \lambda_a(\Gamma)$ having no dependence on t and Γ^p as functions, $(d\lambda_a)/(dt) = (\partial\lambda_a)/(\partial\Gamma^p) = 0$, make it possible [2] to establish the gauge-independence of the the vacuum functional, $Z_{\Phi+\Delta\Phi}(0) = Z_\Phi(0)$, in the particular case of functionally-dependent parameters, $\lambda_a(\Gamma) = \int dt\ \left(s^a \Lambda(t)\right) = \varepsilon_{ab} \int dt\ \left\{\Lambda(t), \Omega^b(t)\right\}_t$ with a certain even-valued potential function $\Lambda(t) = \Lambda(\Gamma(t))$, which is inspired by infinitesimal field-dependent BRST-antiBRST transformations with the parameters [9]

$$\mu_a = \frac{i}{2\hbar}\varepsilon_{ab} \int dt \left\{\Delta\Phi,\ \Omega^b\right\}_t = \frac{i}{2\hbar} \int dt\ \left(s_a \Delta\Phi\right)(t) . \tag{2.40}$$

The gauge-independence of the vacuum functional $Z_\Phi(0)$ implies the gauge-independence of the S-matrix, due to the equivalence theorem [67].

In the case of finite field-dependent transformations with group-like elements $g(\widehat{\Lambda}\overleftarrow{s}_a)$, $\widehat{\Lambda}(\Gamma) = \int dt\, \Lambda(t)$, whose set forms a nonlinear non-Abelian group-like structure,[3] the superdeterminant of a change of variables reads

$$\mathrm{Sdet}\left\{\frac{\delta\left[\left(\Gamma(t')g(\widehat{\Lambda}\overleftarrow{s}_a)\right]\right.}{\delta\Gamma(t'')}\right\} = \exp\left[\Im(\Gamma)\right], \quad \text{where} \quad \Im(\Gamma) = -2\ln\left[1 - \frac{1}{2}\int dt\ \left(s^2\Lambda\right)_t\right] , \tag{2.41}$$

$$d\check{\Gamma} = d\Gamma \exp\left[\frac{i}{\hbar}\left(-i\hbar\Im\right)\right] = d\Gamma \exp\left\{\frac{i}{\hbar}\left[i\hbar \ln\left(1 - \frac{1}{2}\varepsilon_{ab}\int dt \left\{\left\{\Lambda, \Omega^a\right\}_t, \Omega^b\right\}_t\right)^2\right]\right\} , \tag{2.42}$$

with account taken of $\left(s^2\Lambda\right)_t = \varepsilon_{ab}\left\{\left\{\Lambda, \Omega^a\right\}_t, \Omega^b\right\}_t$. In view of the invariance of the quantum action $S_H(\Gamma)$ with respect to (2.38), the change $\Gamma^p(t) \to \check{\Gamma}^p(t) = [\Gamma^p g(\lambda(\Gamma))](t)$ leads to the following transformation of the integrand $\mathcal{I}^\Phi_\Gamma$ in (2.29):

$$\mathcal{I}^\Phi_{\Gamma g(\lambda(\Gamma))} = d\Gamma\ \exp\left[\Im(\Gamma)\right] \exp\left[(i/\hbar)\, S_H\left(\Gamma\left(g\lambda(\Gamma)\right)\right)\right] = d\Gamma\ \exp\left\{(i/\hbar)\left[S_H(\Gamma) - i\hbar\Im(\Gamma)\right]\right\} , \tag{2.43}$$

and thereby implies

$$\mathcal{I}^\Phi_{\Gamma g(\lambda(\Gamma))} = d\Gamma\ \exp\left\{(i/\hbar)\left[S_H(\Gamma) + i\hbar \ln\left(1 - \widehat{\Lambda}\overleftarrow{s}^2/2\right)^2\right]\right\} . \tag{2.44}$$

---

[3]For BRST-antiBRST-closed (in particular, BRST-antiBRST-exact) parameters $\lambda_a(\Gamma) = \widehat{\Lambda}_a\overleftarrow{s}^2$ with $\widehat{\Lambda}_a = \int dt\, \Lambda_a(\Gamma(t), t)$, the subset $g(\widehat{\Lambda}_a\overleftarrow{s}^2)$ forms an Abelian subgroup in $g(\widehat{\Lambda}\overleftarrow{s}_a)$, and thereby in $g(\lambda_a(\Gamma))$; for details, see Footnote 2.

Because of the fact that the Jacobian-induced contribution $i\hbar \ln\left(1-\widehat{\Lambda}\overleftarrow{s}^{2}/2\right)^{2}$ to the action $S_H(\Gamma)$ is a BRST-antiBRST-exact term, it can be compensated by another BRST-antiBRST-exact addition to $S_H(\Gamma)$ related to a change of the gauge-fixing function, $\Phi(t) \to (\Phi+\Delta\Phi)(t)$, made in the original integrand $\mathcal{I}^{\Phi}_{\Gamma}$,

$$i\hbar\,\ln\left(1-\frac{1}{2}\widehat{\Lambda}\overleftarrow{s}^{2}\right)^{2} = -\frac{1}{2}\left(\Delta\widehat{\Phi}\right)\overleftarrow{s}^{2}\ , \quad \text{where} \quad \Delta\widehat{\Phi} = \int dt\, \Delta\Phi(t)\ , \tag{2.45}$$

$$\Longrightarrow \mathcal{I}^{\Phi}_{\Gamma g(\lambda(\Gamma))} = d\Gamma \exp\left\{\frac{i}{\hbar}\left[S_{H,\Phi}(\Gamma) - \frac{1}{2}\left(\Delta\widehat{\Phi}\right)\overleftarrow{s}^{2}\right]\right\} = \mathcal{I}^{\Phi+\Delta\Phi}_{\Gamma}\ . \tag{2.46}$$

The relation of $\Delta\Phi\left(\Gamma(t)|\Lambda\right)$ to the field-dependent parameter $\Lambda\left(\Gamma(t)\right)$ is established by (2.45), also known as the compensation relation for an unknown parameter $\Lambda(\Gamma(t))$, which provides the gauge-independence of the vacuum functional, $Z_{\Phi}(0) = Z_{\Phi+\Delta\Phi}(0)$. An explicit solution of (2.45), satisfying the solvability condition, due to the BRST-antiBRST exactness (up to BRST-antiBRST-exact terms) of both of its sides, is given by

$$\Lambda(\Gamma(t)|\Delta\Phi) = 2\Delta\Phi(t)\left(\Delta\widehat{\Phi}\overleftarrow{s}^{2}\right)^{-1}\left[1-\exp\left(\frac{1}{4i\hbar}\Delta\widehat{\Phi}\overleftarrow{s}^{2}\right)\right] = -\frac{1}{2i\hbar}\Delta\Phi(t)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}\Delta\widehat{\Phi}\overleftarrow{s}^{2}\right)^{n}\ . \tag{2.47}$$

Conversely, having considered the equation (2.45) for an unknown $\Delta\Phi(t)$ with a given $\Lambda(t)$, we obtain

$$\Delta\Phi\left(\Gamma(t)\right) = -2i\hbar\ \Lambda\left(\Gamma(t)\right)\left(\widehat{\Lambda}\left(\Gamma\right)\overleftarrow{s}^{2}\right)^{-1}\ln\left(1-\widehat{\Lambda}\left(\Gamma\right)\overleftarrow{s}^{2}/2\right)^{2}\ . \tag{2.48}$$

Therefore, the field-dependent transformations with the parameters $\lambda_a(\Gamma) = \widehat{\Lambda}\overleftarrow{s}_{a}$,

$$\lambda_a = -\frac{1}{2i\hbar}\left(\Delta\widehat{\Phi}\overleftarrow{s}_{a}\right)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}\Delta\widehat{\Phi}\overleftarrow{s}^{2}\right)^{n}\ ,$$

amount to a precise change of the gauge-fixing function.

In virtue of (2.48), the property (2.46) leads to a so-called modified Ward identity [2] in generalized Hamiltonian formalism, depending on field-dependent parameters, $\lambda_a(\Gamma|\Delta\Phi) = \int dt \Lambda\overleftarrow{s}_{a}$, and thereby also on a finite change of the gauge:

$$\begin{aligned}\Bigg\langle\Bigg\{1+\frac{i}{\hbar}\int dt I_p(t)\Gamma^p(t)\left(\overleftarrow{s}^{a}\lambda_a(\Lambda)+\frac{1}{4}\overleftarrow{s}^{2}\lambda^2(\Lambda)\right)-\frac{1}{4}\left(\frac{i}{\hbar}\right)^{2}\int dt\ dt'\ I_p(t)\Gamma^p(t)\overleftarrow{s}^{a}I_q(t')\Gamma^q(t')\overleftarrow{s}_{a}\lambda^2(\Lambda)\Bigg\}\\ \times\left\{1-\frac{1}{2}\left[\int dt\Lambda(t)\right]\overleftarrow{s}^{2}\right\}^{-2}\Bigg\rangle_{\Phi,I} = 1\,,\end{aligned} \tag{2.49}$$

where the symbol "$\langle\mathcal{A}\rangle_{\Phi,I}$" for any quantity $\mathcal{A} = \mathcal{A}(\Gamma)$ denotes a source-dependent average expectation value corresponding to a gauge $\Phi(\Gamma)$, namely,

$$\langle\mathcal{A}\rangle_{\Phi,I} = Z^{-1}_{\Phi}(I)\int d\Gamma\ \mathcal{A}(\Gamma)\exp\left\{\frac{i}{\hbar}\left[S_{H,\Phi}(\Gamma)+\int dt\, I(t)\Gamma(t)\right]\right\}\ ,\quad \langle 1\rangle_{\Phi,I} = 1\ . \tag{2.50}$$

The property (2.46) implies a relation which describes the gauge-dependence of the generating functional $Z_{\Phi}(I)$,

$$\begin{aligned}Z_{\Phi+\Delta\Phi}(I) = Z_{\Phi}(I)\Bigg\{1+\Bigg\langle\frac{i}{\hbar}\int dt\ I_p(t)\left[(s^a\Gamma^p(t))\lambda_a\left(\Gamma|-\Delta\Phi\right)+\frac{1}{4}(s^2\Gamma^p(t))\lambda^2\left(\Gamma|-\Delta\Phi\right)\right]\\ -\ (-1)^{\varepsilon_q}\left(\frac{i}{2\hbar}\right)^{2}\int dt\ dt' I_q(t')I_p(t)(s^a\Gamma^p(t))(s_a\Gamma^q(t'))\lambda^2\left(\Gamma|-\Delta\Phi\right)\Bigg\rangle\Bigg\}\ ,\end{aligned} \tag{2.51}$$

and extends the result (2.44) to non-vanishing external sources $I_p(t)$.

For constant parameters, $\lambda_a = \mathrm{const}$, the identity (2.49) implies two independent usual Ward identities at the first degree in powers of λ_a, as well as a new (derivative) Ward identity at the second degree in powers of λ_a,

$$\left\langle \int dt\ I_p(t)\Gamma^p(t)\overleftarrow{s}^a \right\rangle_{\Phi,I} = 0\ , \quad \left\langle \int dt\, I_p(t)\Gamma^p(t) \left[\overleftarrow{s}^2 - \overleftarrow{s}^a \left(\frac{i}{\hbar}\right) \int dt'\ I_q(t')\left(\Gamma^q(t')\overleftarrow{s}_a\right)\right] \right\rangle_{\Phi,I} = 0\,. \qquad (2.52)$$

Below, we intend to study the more general case of finite field-dependent BRST-antiBRST transformations in Hamiltonian formalism with arbitrary functional parameters, generally assumed to be functionally-independent, $\lambda_a \not\equiv \int dt\,\Lambda\overleftarrow{s}_a$. It is also intended to study the case of finite field-dependent BRST-antiBRST transformations being linear in functionally-dependent parameters of the form $\lambda_a(\Gamma) = \int dt\ s_a\Lambda(\Gamma(t))$.

2.3 General Gauge Theories in Lagrangian Formalism

The generating functional of Green's functions $Z_F(J)$, depending on external sources J_A, $\varepsilon(J_A) = \varepsilon_A$,

$$Z_F(J) = \int d\mathsf{\Gamma}\ \exp\left\{(i/\hbar)\left[\mathcal{S}_F(\mathsf{\Gamma}) + J_A\phi^A\right]\right\}\ , \quad \mathcal{S}_F = S + \phi^*_{Aa}\pi^{Aa} + \left(\bar{\phi}_A - F_{,A}\right)\lambda^A - (1/2)\,\varepsilon_{ab}\pi^{Aa}F_{,AB}\pi^{Bb}\ , \ (2.53)$$

and the corresponding vacuum functional $Z_F \equiv Z_F(0)$ are defined on the triplectic [68] manifold[4] $\mathcal{M}_{\mathsf{\Gamma}}$ locally parameterized by the coordinates

$$\mathsf{\Gamma}^{\mathsf{p}} = \left(\phi^A, \phi^*_{Aa}, \bar{\phi}_A, \pi^{Aa}, \lambda^A\right)\ , \qquad (2.54)$$

where ϕ^A are the fields of the total configuration space of the BV formalism [21], which is larger in reducible gauge theories, being more general than the theories examined in Section 2.1, and is organized into Sp(2)-symmetric tensors, according to the rules of Sp(2)-covariant Lagrangian quantization [12, 13]. The manifold $\mathcal{M}_{\mathsf{\Gamma}}$ also contains the triplets of antifields ϕ^*_{Aa}, $\bar{\phi}_A$ and auxiliary fields π^{Aa}, λ^A, with the following distribution of Grassmann parity:

$$\varepsilon\left(\phi^A,\ \phi^*_{Aa},\ \bar{\phi}_A,\ \pi^{Aa},\ \lambda^A\right) = \left(\varepsilon_A,\ \varepsilon_A + 1,\ \varepsilon_A,\ \varepsilon_A + 1,\ \varepsilon_A\right).$$

The functional $Z_F(J)$ is determined by an even-valued functional $S = S(\phi, \phi^*, \bar{\phi})$ and by an even-valued gauge-fixing functional $F = F(\phi)$, where S is subject to the generating equations

$$\frac{1}{2}(S,S)^a + V^a S = i\hbar\Delta^a S \Longleftrightarrow \left(\Delta^a + \frac{i}{\hbar}V^a\right)\exp\left(\frac{i}{\hbar}S\right) = 0\ , \qquad (2.55)$$

with the classical action $S_0(A)$ being the boundary condition for S in the case of vanishing antifields, $\phi^*_a = \bar{\phi} = 0$. The extended antibracket $(\bullet,\bullet)^a$ and the operators Δ^a, V^a are given by

$$(\bullet,\bullet)^a = \frac{\delta\bullet}{\delta\phi^A}\frac{\delta\bullet}{\delta\phi^*_{Aa}} - \frac{\delta_r\bullet}{\delta\phi^*_{Aa}}\frac{\delta_l\bullet}{\delta\phi^A}\ , \quad \Delta^a = (-1)^{\varepsilon_A}\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\phi^*_{Aa}}\ , \quad V^a = \varepsilon^{ab}\phi^*_{Ab}\frac{\delta}{\delta\bar{\phi}_A}\ . \qquad (2.56)$$

The classical action is invariant under the infinitesimal gauge transformations (2.4) with the generators $R^i_\alpha(A)$ satisfying the general relations (2.5) of a gauge algebra.

The integrand $\mathcal{I}^F_{\mathsf{\Gamma}} = d\mathsf{\Gamma}\exp\left[(i/\hbar)\,\mathcal{S}_F(\mathsf{\Gamma})\right]$ is invariant under the global infinitesimal BRST-antiBRST transformations (2.57), with the corresponding generators s^a being different from s^a of Subsections 2.1, 2.2,

$$\delta\mathsf{\Gamma}^{\mathsf{p}} = \left(\mathsf{s}^a\mathsf{\Gamma}^{\mathsf{p}}\right)\mu_a = \mathsf{\Gamma}^{\mathsf{p}}\overleftarrow{\mathsf{s}}^a\mu_a = \left(\pi^{Aa},\ \delta^a_b S_{,A}(-1)^{\varepsilon_A}\ ,\ \varepsilon^{ab}\phi^*_{Ab}(-1)^{\varepsilon_A+1}\ ,\ \varepsilon^{ab}\lambda^A,\ 0\right)\mu_a\ , \qquad (2.57)$$

where the invariance at the first order in μ_a is established by using the generating equations (2.55).

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[4]Amongst the ingredients of [68], we only use differential operations in local coordinates, and therefore our description of triplectic geometry reduces to a description of BRST-antiBRST quantization [12, 13, 14].

Despite the fact that the generators s^a do not obey, $\mathsf{s}^a\mathsf{s}^b + \mathsf{s}^b\mathsf{s}^a \neq 0$, the BRST-antiBRST algebra in the sector of the antifields ϕ^*_{Aa}, $\bar{\phi}_A$, the mentioned infinitesimal invariance is sufficient to determine *finite BRST-antiBRST transformations*, $\Gamma^p \to \Gamma^p + \Delta\Gamma^p$, with anticommuting parameters λ_a, $a = 1,2$, introduced in [1] according to

$$\mathcal{I}^F_{\Gamma+\Delta\Gamma} = \mathcal{I}^F_\Gamma \ , \quad \Delta\Gamma^p \frac{\overleftarrow{\partial}}{\partial\lambda_a}\bigg|_{\lambda=0} = \Gamma^p \overleftarrow{\mathsf{s}}^a \quad \text{and} \quad \Delta\Gamma^p \frac{\overleftarrow{\partial}}{\partial\lambda_b}\frac{\overleftarrow{\partial}}{\partial\lambda_a} = \frac{1}{2}\varepsilon^{ab}\Gamma^p\overleftarrow{\mathsf{s}}^2 \ , \quad \text{where} \quad \mathsf{s}^2 = \mathsf{s}_a\mathsf{s}^a = \overleftarrow{\mathsf{s}}^2 = \overleftarrow{\mathsf{s}}^a\overleftarrow{\mathsf{s}}_a \ . \tag{2.58}$$

The finite BRST-antiBRST transformations for the integrand $\mathcal{I}^F_\Gamma$ in a general gauge theory are established, once again, by solving the functional equation $G(\Gamma + \Delta\Gamma) = G(\Gamma)$ for any regular functional $G(\Gamma)$ defined in $\mathcal{M}_\Gamma$ and subject to infinitesimal BRST-antiBRST invariance, $G\overleftarrow{\mathsf{s}}^a = 0$, which may be considered as the integrability condition for the above functional equation. The resulting finite BRST-antiBRST transformations are given by

$$\Delta\Gamma^p = \Gamma^p\left(\overleftarrow{\mathsf{s}}^a\lambda_a + \frac{1}{4}\overleftarrow{\mathsf{s}}^2\lambda^2\right) \ , \tag{2.59}$$

or, equivalently, in a group-like form

$$\Gamma'^p = \Gamma^p\left(1 + \overleftarrow{\mathsf{s}}^a\lambda_a + \frac{1}{4}\overleftarrow{\mathsf{s}}^2\lambda^2\right) = \Gamma^p \exp\left(\overleftarrow{\mathsf{s}}^a\lambda_a\right) \equiv \Gamma^p\mathsf{g}(\lambda) \ , \tag{2.60}$$

so that there holds the exact relation

$$\mathcal{I}^F_{\Gamma\mathsf{g}(\lambda)} = \mathcal{I}^F_\Gamma \ , \tag{2.61}$$

considering that the functional S meets the generating equations (2.55). To establish the relation (2.61), we need to take into account the change of the integration measure under the global finite transformations (corresponding to $\lambda_a = \text{const}$) and the respective change of the functional $\mathcal{S}_F(\Gamma)$ in (2.53), according to the rules [3]

$$d\Gamma' = d\Gamma\ \mathrm{Sdet}\left[\frac{\delta\left(\Gamma\mathsf{g}(\lambda)\right)}{\delta\Gamma}\right] = d\Gamma \exp\left[-\left(\Delta^a S\right)\lambda_a - \frac{1}{4}\left(\Delta^a S\right)\overleftarrow{\mathsf{s}}_a\lambda^2\right] \ , \tag{2.62}$$

$$\exp\left[\frac{i}{\hbar}\mathcal{S}_F\left(\Gamma'\right)\right] = \exp\left\{\frac{i}{\hbar}\left[\mathcal{S}_F\left(\Gamma\right) + \mathcal{S}_F\left(\Gamma\right)\overleftarrow{\mathsf{s}}^a\lambda_a + \frac{1}{4}\mathcal{S}_F\left(\Gamma\right)\overleftarrow{\mathsf{s}}^2\lambda^2\right]\right\} \ , \tag{2.63}$$

so that, due to the relations

$$\mathcal{S}_F\overleftarrow{\mathsf{s}}^a = -\frac{1}{2}(S,S)^a - V^a S \,, \tag{2.64}$$

implied by (2.55), the finite BRST-antiBRST invariance (2.61) of $\mathcal{I}^F_\Gamma$ does indeed take place.

The set of elements $\{\mathsf{g}(\lambda)\} = \{\exp(\overleftarrow{\mathsf{s}}^a\lambda_a)\}$, in contrast to the respective sets of finite BRST-antiBRST transformations (2.12), (2.37) in Yang–Mills theories and first-class constraint dynamical systems, does not form a supergroup with respect to multiplication, denoted by the symbol "$\cdot$", being an associative composition law. Indeed, for any elements $\mathsf{g}(\lambda_{(1)})$, $\mathsf{g}(\lambda_{(2)})$ their composition is given by[5]

$$\mathsf{g}(\lambda_{(1)})\cdot\mathsf{g}(\lambda_{(2)}) = \mathsf{g}(\lambda_{(1)} + \lambda_{(2)}) + \mathrm{dev}(\lambda_{(1)}, \lambda_{(2)}) \ , \tag{2.65}$$

$$\begin{aligned}\mathrm{dev}(\lambda_{(1)}, \lambda_{(2)}) = &-\frac{1}{2}\overleftarrow{s}^b\overleftarrow{s}^a\left[\lambda_{(2)b}\lambda_{(1)a} - \lambda_{(1)b}\lambda_{(2)a}\right] \\ &+ \frac{1}{4}\left[\overleftarrow{s}^2\overleftarrow{s}^a\lambda^2_{(2)}\lambda_{(1)a} + \overleftarrow{s}^a\overleftarrow{s}^2\lambda^2_{(1)}\lambda_{(2)a}\right] + \frac{1}{16}\overleftarrow{s}^2\overleftarrow{s}^2\lambda^2_{(2)}\lambda^2_{(1)} \neq 0 \ ,\end{aligned} \tag{2.66}$$

and therefore contains non-vanishing operator structures, $\overleftarrow{\mathsf{s}}^2\overleftarrow{\mathsf{s}}^a$, $\overleftarrow{\mathsf{s}}^a\overleftarrow{\mathsf{s}}^2$, $\overleftarrow{\mathsf{s}}^2\overleftarrow{\mathsf{s}}^2$, which are absent from a group element $\mathsf{g}\left(\lambda\right)$. Notice that the non-vanishing deviation $\mathrm{dev}(\lambda_{(1)}, \lambda_{(2)})$ of the action of $\{\mathsf{g}(\lambda)\}$ from that of an

[5] In case the parameters $\lambda^a_{(i)}$, $i = 1,2$, belong to a vector space with some anticommuting basis elements ν^a, $\mathrm{gh}\left(\nu^a\right) = (-1)^{a+1}$, namely, $\lambda^a_{(i)} = c_{(i)}\left(a\right)\cdot\nu^a$, with certain c-numbers $c_{(i)}\left(a\right)$ and no summation over a, it follows that $\lambda^2_{(i)}\lambda_{(j)a} = \lambda^2_{(i)}\lambda^2_{(j)} \equiv 0$; however, the deviation $\mathrm{dev}(\lambda_{(1)}, \lambda_{(2)})$ remains non-vanishing, $\mathrm{dev}(\lambda_{(1)}, \lambda_{(2)}) = -\frac{1}{2}\overleftarrow{s}^b\overleftarrow{s}^a\left[c_{(2)}\left(b\right)c_{(1)}\left(a\right) - c_{(1)}\left(b\right)c_{(2)}\left(a\right)\right]\nu_b\nu_a$, which is readily seen in components: $\mathrm{dev}(\lambda_{(1)}, \lambda_{(2)}) = (1/2)\left(\overleftarrow{s}^1\overleftarrow{s}^2 + \overleftarrow{s}^2\overleftarrow{s}^1\right)\left[c_{(1)}\left(1\right)c_{(2)}\left(2\right) - c_{(1)}\left(2\right)c_{(2)}\left(1\right)\right]\nu_1\nu_2 \neq 0$.

Abelian two-parameter supergroup is not symmetric with respect to the permutation of the arguments, $\lambda_1 \leftrightarrow \lambda_2$: $\mathrm{dev}(\lambda_{(1)},\lambda_{(2)}) \neq \mathrm{dev}(\lambda_{(2)},\lambda_{(1)})$. This implies that the commutator of any $\mathsf{g}(\lambda_{(1)})$, $\mathsf{g}(\lambda_{(2)})$ in the set $\{\mathsf{g}(\lambda)\}$ is non-vanishing:

$$[\mathsf{g}(\lambda_{(1)}),\mathsf{g}(\lambda_{(2)})] \equiv \mathsf{g}(\lambda_{(1)})\cdot\mathsf{g}(\lambda_{(2)}) - \mathsf{g}(\lambda_{(2)})\cdot\mathsf{g}(\lambda_{(1)}) \neq 0\,.$$

At the same time, the set $\{\mathsf{g}(\lambda)\}$, being considered as right-hand transformations realized on regular functionals in $M_{\mathsf{\Gamma}}$ restricted to $\tilde{G} = \tilde{G}(\phi,\pi,\lambda)$, $\tilde{G} = G(\mathsf{\Gamma})|_{\phi^*=\bar{\phi}=0}$, turns into an Abelian supergroup $\{\tilde{\mathsf{g}}(\lambda)\}$ with the elements

$$\{\tilde{\mathsf{g}}(\lambda)\} = \left\{\tilde{\mathsf{g}}(\lambda) \in \{\mathsf{g}(\lambda)\} | \tilde{\mathsf{g}}(\lambda) = \exp(\overleftarrow{U}{}^a\lambda_a)\,, \quad \overleftarrow{U}{}^a = \overleftarrow{\mathsf{s}}{}^a\big|_{\phi,\pi,\lambda}\right\}\,, \tag{2.67}$$

where the operators $\overleftarrow{U}{}^a$ are anticommuting and thereby nilpotent [4], namely,

$$\overleftarrow{U}{}^a = \frac{\overleftarrow{\delta}}{\delta\phi^A}\pi^{Aa} + \varepsilon^{ab}\frac{\overleftarrow{\delta}}{\delta\pi^{Ab}}\lambda^A\,, \quad \overleftarrow{U}{}^a\overleftarrow{U}{}^b + \overleftarrow{U}{}^b\overleftarrow{U}{}^a = 0\,, \quad \overleftarrow{U}{}^a\overleftarrow{U}{}^b\overleftarrow{U}{}^c = 0\,. \tag{2.68}$$

Indeed, due to the nilpotency of $\overleftarrow{U}{}^a$, it follows that

$$\tilde{\mathsf{g}}(\lambda_{(1)})\cdot\tilde{\mathsf{g}}(\lambda_{(2)}) = \tilde{\mathsf{g}}(\lambda_{(1)}+\lambda_{(2)}) + \mathrm{dev}(\lambda_{(1)},\lambda_{(2)})\,, \tag{2.69}$$

$$\mathrm{dev}(\lambda_{(1)},\lambda_{(2)}) = -\frac{1}{2}\overleftarrow{U}{}^b\overleftarrow{U}{}^a\left[\lambda_{(2)b}\lambda_{(1)a} - \lambda_{(1)b}\lambda_{(2)a}\right] = 0\,, \tag{2.70}$$

since

$$\overleftarrow{U}{}^b\overleftarrow{U}{}^a\left[\lambda_{(2)b}\lambda_{(1)a} - \lambda_{(1)b}\lambda_{(2)a}\right] = -\frac{1}{2}\overleftarrow{U}{}^2\left[\lambda_{(2)a}\lambda^a_{(1)} - \lambda_{(1)a}\lambda^a_{(2)}\right] = -\frac{1}{2}\overleftarrow{U}{}^2\left[\lambda_{(2)a}\lambda^a_{(1)} - \lambda_{(2)a}\lambda^a_{(1)}\right] \equiv 0\,, \tag{2.71}$$

which proves the Abelian nature of the supergroup $\{\tilde{\mathsf{g}}(\lambda)\}$, namely, $\tilde{\mathsf{g}}(\lambda_{(1)})\cdot\tilde{\mathsf{g}}(\lambda_{(2)}) = \tilde{\mathsf{g}}(\lambda_{(1)}+\lambda_{(2)}) = \tilde{\mathsf{g}}(\lambda_{(2)})\cdot\tilde{\mathsf{g}}(\lambda_{(1)})$.

For finite field-dependent transformations, it has been shown [3, 4] that in the case of functionally-dependent parameters $\lambda_a = \Lambda\overleftarrow{U}{}_a$ with an even-valued potential $\Lambda = \Lambda(\phi,\pi,\lambda)$, inspired by infinitesimal field-dependent BRST-antiBRST transformations with the parameters [1, 3]

$$\mu_a(\phi,\pi,\lambda|\Delta F) = -\frac{i}{2\hbar}\varepsilon_{ab}\left(\Delta F\right)_{,A}\pi^{Ab} = -\frac{i}{2\hbar}\varepsilon_{ab}\Delta F\overleftarrow{U}{}^b\,, \tag{2.72}$$

there holds the gauge-independence of the vacuum functional: $Z_{F+\Delta F}(0) = Z_F(0)$. Indeed, a finite transformation with a group-like element $\tilde{\mathsf{g}}(\Lambda\overleftarrow{U}{}_a)$ leads to the superdeterminant of a change of variables $\mathsf{\Gamma}^{\mathsf{p}} \to \mathsf{\Gamma}^{\mathsf{p}}\tilde{\mathsf{g}}(\Lambda\overleftarrow{U}{}_a)$ and implies the corresponding change of the integration measure given by [3, 4]:

$$\mathrm{Sdet}\left\{\frac{\delta\left[\mathsf{\Gamma}\breve{g}(\Lambda\overleftarrow{U}{}_a)\right]}{\delta\mathsf{\Gamma}}\right\} = \exp\left[-\left(\Delta^a S\right)\lambda_a - \frac{1}{4}\left(\Delta^a S\right)\overleftarrow{U}{}_a\lambda^2\right]\exp\left[\ln\left(1-\frac{1}{2}\Lambda\overleftarrow{U}{}^2\right)^{-2}\right]\,, \tag{2.73}$$

$$d\mathsf{\Gamma}' = d\mathsf{\Gamma}\,\mathrm{Sdet}\left\{\frac{\delta\left[\mathsf{\Gamma}\breve{g}(\Lambda\overleftarrow{U}{}_a)\right]}{\delta\mathsf{\Gamma}}\right\} = d\mathsf{\Gamma}\;\exp\left\{\frac{i}{\hbar}\left[i\hbar\left(\Delta^a S\right)\lambda_a + \frac{i\hbar}{4}\left(\Delta^a S\right)\overleftarrow{U}{}_a\lambda^2 + i\hbar\ln\left(1-\frac{1}{2}\Lambda\overleftarrow{U}{}^2\right)^2\right]\right\}\,. \tag{2.74}$$

Using the Jacobian (2.73), the transformation of the action $\mathcal{S}_F$ according to (2.64), the equations (2.55) with their consequence resulting from applying $\overleftarrow{\mathsf{s}}{}^a$, and the BRST-antiBRST exactness of the term $F\overleftarrow{U}{}^2$, we arrive at [3, 4]

$$\begin{aligned} Z_F &\overset{\mathsf{\Gamma}\to\mathsf{\Gamma}'}{=} \int d\mathsf{\Gamma}\;\exp\left\{\frac{i}{\hbar}\left[\mathcal{S}_F + \left(\mathcal{S}_F\overleftarrow{\mathsf{s}}{}^a + i\hbar\Delta^a S\right)\lambda_a + \frac{1}{4}\left(\mathcal{S}_F\overleftarrow{\mathsf{s}}{}^2 + i\hbar\Delta^a S\overleftarrow{\mathsf{s}}{}_a\right)\lambda^2 + i\hbar\ln\left(1-\frac{1}{2}\Lambda\overleftarrow{U}{}^2\right)^2\right]\right\} \\ &= \int d\mathsf{\Gamma}\;\exp\left\{\frac{i}{\hbar}\left[\mathcal{S}_{F+\Delta F} + i\hbar\ln\left(1-\frac{1}{2}\Lambda\overleftarrow{\mathsf{s}}{}^2\right)^2 + \frac{1}{2}\Delta F\overleftarrow{U}{}^2\right]\right\}\,. \end{aligned} \tag{2.75}$$

The coincidence of the vacuum functionals Z_F and $Z_{F+\Delta F}$, evaluated for the respective even-valued functionals F and $F+\Delta F$, is valid, together with a compensation equation for an unknown even-valued functional Λ:

$$i\hbar \ln\left(1-\frac{1}{2}\Lambda\overleftarrow{U}^2\right)^2 = -\frac{1}{2}\Delta F\overleftarrow{U}^2 . \tag{2.76}$$

An explicit solution of (2.76) satisfying the solvability condition (that both sides should be BRST-antiBRST-exact) has the usual form – up to $\overleftarrow{U}^a$-exact terms – identical with (2.22) for the similar equations (2.20), (2.20) in Yang–Mills theories and first-class constraint dynamical systems:

$$\Lambda\left(\phi,\pi,\lambda|\Delta F\right) = 2\Delta F\left(\Delta F\overleftarrow{U}^2\right)^{-1}\left[\exp\left(-\frac{1}{4i\hbar}\Delta F\overleftarrow{U}^2\right)-1\right] = \frac{1}{2i\hbar}\Delta F\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(-\frac{1}{4i\hbar}\Delta F\overleftarrow{U}^2\right)^n . \tag{2.77}$$

Conversely, the equation (2.76) examined for a certain unknown change ΔF of the gauge-fixing functional for a given functional $\Lambda\left(\phi,\pi,\lambda\right)$ has the following solution, with accuracy up to $\overleftarrow{U}^a$-exact terms:

$$\Delta F\left(\phi,\pi,\lambda\right) = -2i\hbar\,\Lambda\left(\phi,\pi,\lambda\right)\left(\Lambda\left(\phi,\pi,\lambda\right)\overleftarrow{U}^2\right)^{-1}\ln\left(1-\Lambda\left(\phi,\pi,\lambda\right)\overleftarrow{U}^2/2\right)^2 . \tag{2.78}$$

Field-dependent transformations with the functional-dependent parameters $\lambda_a = \Lambda\overleftarrow{U}_a$ given by

$$\lambda_a = \frac{1}{2i\hbar}\left(\Delta F\overleftarrow{U}_a\right)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}\Delta F\overleftarrow{U}^2\right)^n$$

amount to a precise change of the gauge-fixing functional in a general gauge theory.

It has been shown [3] that the relation $Z_F = Z_{F+\Delta F}$ in (2.75) leads to the presence of a modified Ward identity,

$$\left\langle\left\{1+\frac{i}{\hbar}J_A\phi^A\left[\overleftarrow{U}^a\lambda_a(\Lambda)+\frac{1}{4}\overleftarrow{U}^2\lambda^2(\Lambda)\right]-\frac{1}{4}\left(\frac{i}{\hbar}\right)^2 J_A\phi^A\overleftarrow{U}^a J_B(\phi^B)\overleftarrow{U}_a\lambda^2(\Lambda)\right\}\left(1-\frac{1}{2}\Lambda\overleftarrow{U}^2\right)^{-2}\right\rangle_{F,J} = 1 , \tag{2.79}$$

and allows one to study the gauge dependence of $Z_F(J)$ in (2.53) for a finite change of the gauge $F\to F+\Delta F$,

$$\begin{aligned} Z_{F+\Delta F}(J) = Z_F(J)\Bigg\{1+\Bigg\langle\frac{i}{\hbar}J_A\phi^A\left[\overleftarrow{U}^a\lambda_a\left(\Gamma|-\Delta F\right)+\frac{1}{4}\overleftarrow{U}^2\lambda^2\left(\Gamma|-\Delta F\right)\right] \\ -\ (-1)^{\varepsilon_B}\left(\frac{i}{2\hbar}\right)^2 J_B J_A\left(\phi^A\overleftarrow{U}^a\right)\left(\phi^B\overleftarrow{U}_a\right)\lambda^2\left(\Gamma|-\Delta F\right)\Bigg\rangle_{F,J}\Bigg\} , \end{aligned} \tag{2.80}$$

where the symbol "$\langle\mathcal{A}\rangle_{F,J}$" for a quantity $\mathcal{A}=\mathcal{A}(\Gamma)$ stands for a source-dependent average expectation value corresponding to a gauge-fixing $F(\phi,\pi,\lambda)$:

$$\langle\mathcal{A}\rangle_{F,J} = Z_F^{-1}(J)\int d\Gamma\ \mathcal{A}\left(\Gamma\right)\exp\left\{\frac{i}{\hbar}\left[\mathcal{S}_F\left(\Gamma\right)+J_A\phi^A\right]\right\} ,\quad \text{where}\quad \langle 1\rangle_{F,J}=1 . \tag{2.81}$$

In the case of constant λ_a, the modified Ward identity (2.79) contains an Sp(2)-doublet of the usual Ward identities at the first order in λ_a and a derivative identity at the second order in λ_a:

$$J_A\left\langle\phi^A\overleftarrow{U}^a\right\rangle_{F,J} = 0 ,\quad \left\langle J_A\left[\phi^A\overleftarrow{U}^2+(i/\hbar)\,\varepsilon_{ab}\phi^A\overleftarrow{U}^a J_B(\phi^B\overleftarrow{U}^b)\right]\right\rangle_{F,J} = 0 . \tag{2.82}$$

In the case of first-rank gauge theories with a closed gauge algebra, $M^{ij}_{\alpha\beta}=0$, in (2.5), provided that the solution to the generating equations (2.55) is linear in the antifields ϕ^*_{Aa}, $\bar{\phi}_A$, the representation (2.53) for $Z_F(J)$ reduces to (2.1), with the action $S_F(\phi)$ being identical to (2.2) in irreducible gauge theories (2.6), (2.7), in particular, Yang–Mills theories (2.8). The study of finite (field-dependent) BRST-antiBRST transformations and their consequences to the quantum properties of a theory is then reduced to the results of Section 2.1. Below, we intend to study the case of finite field-dependent BRST-antiBRST transformations for general gauge theories with arbitrary functional parameters, generally assumed to be functionally-independent, $\lambda_a \not\equiv s_a\Lambda$.

3 Linearized Finite BRST-antiBRST Transformations

In this section, we examine the calculation of the Jacobian for the linear part of finite field-dependent BRST-antiBRST transformations, i.e., the part being linear in functionally-dependent parameters of the form $\lambda_a(\phi) = s_a\Lambda(\phi)$ and $\lambda_a(\Gamma) = \int dt\, s_a\Lambda(\Gamma)$, respectively, in Yang–Mills theories and arbitrary dynamical systems with first-class constraints. We shall carry out the explicit calculations in the Yang–Mills case and then translate the resulting Jacobian to the case of dynamical systems in question, using the anticommutativity, $s^a s^b + s^b s^a = 0$, of the corresponding generators s^a and the invariance of the functional integration measure, $d\phi$ and $d\Gamma$, under global BRST-antiBRST transformations, $\lambda_a = \text{const}$, in both these cases.

3.1 Yang–Mills Theories

In the Yang–Mills case, the linear part of finite field-dependent BRST-antiBRST transformations in question has the form

$$\phi^A \to \phi'^A = \phi^A + \Delta\phi^A \ , \quad \text{where} \quad \Delta\phi^A = \left(s^a\phi^A\right)\lambda_a = X^{Aa}\lambda_a \ , \quad \lambda_a = s_a\Lambda \ . \tag{3.1}$$

Let us examine the even matrix $M = \|M^A_B\|$

$$\frac{\delta\left(\Delta\phi^A\right)}{\delta\phi^B} = M^A_B \ , \quad \varepsilon\left(M^A_B\right) = \varepsilon_A + \varepsilon_B \ ,$$

and the corresponding Jacobian $\exp(\Im)$

$$\Im = \mathrm{Str}\ln\left(\mathbb{I} + M\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(M^n\right) \ , \quad \text{where} \quad \mathrm{Str}\left(M^n\right) = \left(M^n\right)^A_A(-1)^{\varepsilon_A} \ , \quad \mathbb{I}^A_B = \delta^A_B \ .$$

Explicitly, the matrix M^A_B is given by the sum of two even matrices:

$$\begin{aligned} M^A_B &= X^{Aa}\frac{\delta\lambda_a}{\delta\phi^B} + \frac{\delta X^{Aa}}{\delta\phi^B}\lambda_a(-1)^{\varepsilon_B} \equiv P^A_B + Q^A_B \ , \\ P^A_B &= X^{Aa}\frac{\delta\lambda_a}{\delta\phi^B} \ , \quad Q^A_B = \frac{\delta X^{Aa}}{\delta\phi^B}\lambda_a(-1)^{\varepsilon_B} \ . \end{aligned} \tag{3.2}$$

Further considerations are based on the following statements, established in our previous work [1], using the properties

$$X^{Aa}_{,A} = 0 \ , \quad s^b s^a\phi^A = s^b X^{Aa} = \varepsilon^{ab}Y^A \ ,$$

which take place in Yang–Mills theories, and the supertrace property

$$\mathrm{Str}\left(AB\right) = \mathrm{Str}\left(BA\right) \ , \tag{3.3}$$

which takes place for arbitrary even matrices A and B:

Proposition 1 *The matrices (3.2) with arbitrary odd-valued $\lambda_a \not\equiv s_a\Lambda$ obey the properties*

$$\mathrm{Str}\left(P+Q\right)^n = \mathrm{Str}\left(P^n + nP^{n-1}Q + C^2_n P^{n-2}Q^2\right) \ , \quad \text{where} \quad C^k_n = \frac{n!}{k!\,(n-k)!} \ , \quad n = 2,3 \ , \tag{3.4}$$

$$\mathrm{Str}\left(Q\right) = 0 \ , \quad \mathrm{Str}\left(Q^2\right) = 2\mathrm{Str}\left(R\right) \ , \quad \text{where} \quad R^A_B \equiv -\frac{1}{2}\lambda^2\frac{\delta Y^A}{\delta\phi^B} \ . \tag{3.5}$$

Proposition 2 *Let us suppose that the condition $\lambda_a = s_a\Lambda$ is fulfilled. Then there hold the properties*[6]

$$P^2 = f \cdot P \Longrightarrow P^n = f \cdot P^{n-1} \ , \ \text{where } s^a\lambda_b = \delta^a_b f \ \Longrightarrow f = -\frac{1}{2}s^2\Lambda = -\frac{1}{2}\mathrm{Str}\,(P) \ , \tag{3.6}$$

$$QP = (1+f)\cdot Q_2 \ , \ \text{where } \left(Q_2\right)^A_B \equiv \lambda_a Y^A \frac{\delta\lambda^a}{\delta\phi^B}\,(-1)^{\varepsilon_A+1} \ , \tag{3.7}$$

$$\mathrm{Str}\,(P+Q)^n = \mathrm{Str}\left(P^n + nP^{n-1}Q^1 + nP^{n-2}Q^2 + K_nP^{n-3}QPQ\right), \ \text{where } K_n \equiv C_n^2 - C_n^1 = \frac{n\,(n-3)}{2} \ , \ n \geq 4 \ . \tag{3.8}$$

Note: the equality (3.4) is entirely due to the Grassmann parity of P, Q and the character of dependence of these matrices on λ_a; the property $\mathrm{Str}\,(Q)=0$ in (3.5) translates to the invariance, $d\phi' = d\phi$, of functional integration measure under global BRST-antiBRST transformations $\delta\phi^A = \left(s^a\phi^A\right)\lambda_a$, $\lambda_a = \text{const}$, while the property $\mathrm{Str}\left(Q^2\right) = 2\mathrm{Str}\,(R)$ is implied by the anticommutativity, $s^as^b + s^bs^a = 0$, of the generators s^a, as well as by the above-mentioned invariance of functional integration measure, encoded in $\mathrm{Str}\,(Q) = 0$; the properties (3.6), (3.7), substantially related to $\lambda_a = s_a\Lambda$, are implied by the anticommutativity of the generators s^a; the combinatorial coefficient K_n in (3.8) corresponds to the decomposition of the binomial coefficient C_n^2 into two parts: C_n^1 and $K_n = C_n^2 - C_n^1$; in fact, the coefficient K_n is the number of monomials in $(P+Q)^n$ for $n \geq 4$ that contain two matrices Q and cannot be transformed by cyclic permutations under the symbol Str of supertrace to the form $\mathrm{Str}(P^{n-2}Q^2)$ by using (3.3); in virtue of the contraction property $P^n = f \cdot P^{n-1}$ in (3.6) the supertrace of all such monomials is equal to $\mathrm{Str}\left(P^{n-3}QPQ\right)$; Propositions 1, 2 will be proved independently in Subsection 4.1, which deals with the case of arbitrary parameters $\lambda_a \not\equiv s_a\Lambda$.

From the above properties (3.3)–(3.8), it follows that the quantity $\Im$ takes the form (see Appendix A)

$$\Im = -2\ln(1+f) + \Re \ , \quad \Re = -\mathrm{Str}\left[R + Q_2 + (1/2)\,Q_2^2 - (1+f)\,QQ_2\right] \ , \tag{3.9}$$

where

$$f = -\frac{1}{2}s^2\Lambda = \frac{1}{2}s^as_a\Lambda \ .$$

Substituting the explicit form of the matrices (3.2), (3.5), (3.7), we have

$$\Re = \frac{1}{2}\,(-1)^{\varepsilon_A}\left[\frac{\delta\left(Y^A\lambda^2\right)}{\delta\phi^A} - \frac{1}{4}Y^A\frac{\delta\lambda^2}{\delta\phi^B}Y^B\frac{\delta\lambda^2}{\delta\phi^A} - (1+f)\,\frac{\delta X^{Aa}}{\delta\phi^B}Y^B\lambda_a\frac{\delta\lambda^2}{\delta\phi^A}\right] \ . \tag{3.10}$$

Using the relations $X^{Aa} = s^a\phi^A$, $Y^A = -(1/2)\,s^2\phi^A$ and bearing in mind that $\lambda_a = s_a\Lambda$, one can represent the contribution (3.10) in terms of BRST-antiBRST variations:

$$\Re = -\frac{1}{4}\,(-1)^{\varepsilon_A}\left\{\left[\left(s^2\phi^A\right)\lambda^2\right]_{,A} + \frac{1}{8}\left(s^2\phi^A\right)\lambda^2_{,B}\left(s^2\phi^B\right)\lambda^2_{,A} - \left(1 - \frac{1}{2}s^2\Lambda\right)\left(s^a\phi^A\right)_{,B}\left(s^2\phi^B\right)\lambda_a\lambda^2_{,A}\right\} \ . \tag{3.11}$$

Let us now examine the transformation of the integrand

$$d\phi\exp\left[(i/\hbar)\,S_F\,(\phi)\right]$$

in the Yang–Mills path integral under the linearized finite BRST-antiBRST transformations (3.1):

$$\begin{aligned} d\phi\exp\left[(i/\hbar)\,S_F\,(\phi)\right]\big|_{\phi\to\phi'} &= d\phi\ J\,(\phi)\exp\left[(i/\hbar)\,S_F\,(\phi+\Delta\phi)\right] = d\phi\ \exp\left\{(i/\hbar)\left[S_F\,(\phi+\Delta\phi) - i\hbar\Im\,(\phi)\right]\right\} \\ &= d\phi\ \exp\left\{(i/\hbar)\left[S_F\,(\phi+\Delta\phi) + i\hbar\ln\left[1-(1/2)\,s^2\Lambda\,(\phi)\right]^2 - i\hbar\Re\,(\phi)\right]\right\} \ , \end{aligned} \tag{3.12}$$

[6]Further on, we will use different forms of the same matrices: $Q^A_B = X^{Aa}_{,B}\lambda_a\,(-1)^{\varepsilon_B} = \lambda_aX^{Aa}_{,B}\,(-1)^{\varepsilon_A+1}$, $(Q_2)^A_B = \lambda_aY^A\lambda^a_{,B}\,(-1)^{\varepsilon_A+1} = -(1/2)\,Y^A\lambda^2_{,B}$.

where

$$
\begin{aligned}
S_F(\phi+\Delta\phi) &= S_F(\phi) + \frac{\delta S_F}{\delta\phi^A}(\phi)\,\Delta\phi^A + \frac{1}{2}\frac{\delta^2 S_F}{\delta\phi^A\delta\phi^B}(\phi)\,\Delta\phi^B\Delta\phi^A \\
&= S_F(\phi) + \frac{1}{2}(-1)^{\varepsilon_A}\,\frac{\delta^2 S_F}{\delta\phi^A\delta\phi^B}(\phi)\,X^{Bb}(\phi)\,X^{Aa}(\phi)\,\lambda_a(\phi)\,\lambda_b(\phi)\ \ .
\end{aligned}
\tag{3.13}
$$

Here, the first order of expansion in $\Delta\phi^A = X^{Aa}\lambda_a$ drops out due to the invariance property $s^a S_F = 0$,

$$
s^a S_F = \frac{\delta S_F}{\delta\phi^A}X^{Aa} = 0\ . \tag{3.14}
$$

Then, differentiating the above relation,

$$
\frac{\delta S_F}{\delta\phi^A}\frac{\delta X^{Aa}}{\delta\phi^B} + \frac{\delta^2 S_F}{\delta\phi^B\delta\phi^A}X^{Aa}(-1)^{\varepsilon_B} = 0\ , \tag{3.15}
$$

multiplying the result from the right by the quantity $X^{Bb}\lambda_b\lambda_a$ and using the property $X^{Aa}_{,B}X^{Bb} = \varepsilon^{ab}Y^A$, we obtain

$$
S_F(\phi+\Delta\phi) = S_F(\phi) + \frac{1}{2}\frac{\delta S_F}{\delta\phi^A}(\phi)\,Y^A(\phi)\,\lambda^2(\phi) = S_F(\phi) - \frac{1}{4}\frac{\delta S_F}{\delta\phi^A}(\phi)\left(s^2\phi^A\right)\lambda^2(\phi)\ \ , \tag{3.16}
$$

which implies the following transformation of the integrand:

$$
d\phi\exp\left[\frac{i}{\hbar}S_F(\phi)\right]\bigg|_{\phi\to\phi'} = d\phi\exp\left\{\frac{i}{\hbar}\left[S_F(\phi) + i\hbar\ln\left(1-\frac{1}{2}s^2\Lambda(\phi)\right)^2 + S_F^{\mathrm{add}}(\phi)\right]\right\} \equiv d\phi\exp\left[\frac{i}{\hbar}S'(\phi)\right]\ \ , \tag{3.17}
$$

where

$$
S_F^{\mathrm{add}} = \frac{1}{2}\frac{\delta S_F}{\delta\phi^A}Y^A\lambda^2 - i\hbar\Re\ . \tag{3.18}
$$

The above expression is obviously not BRST-antiBRST-invariant: $s^a S_F^{\mathrm{add}} \not\equiv 0$. As a consequence, the corresponding quantum action $S'(\phi)$ fails to be BRST-antiBRST-invariant, $s^a S' \not\equiv 0$, and therefore it does not amount to an exact change of the gauge-fixing functional:

$$
S'(\phi) \neq S_0(A) - \frac{1}{2}s^2 F'(\phi)\ \ . \tag{3.19}
$$

Finally, it should be noted that the integrand fails to be invariant under global linearized finite BRST-antiBRST transformations, $\lambda_a = \mathrm{const}$:

$$
d\phi\exp\left[\frac{i}{\hbar}S_F(\phi)\right]\bigg|_{\phi\to\phi'} - d\phi\exp\left[\frac{i}{\hbar}S_F(\phi)\right] = d\phi\exp\left[\frac{i}{\hbar}S_F(\phi)\right]\left\{\exp\left[\frac{i}{\hbar}S_F^{\mathrm{add}}(\phi)\right] - 1\right\} \neq 0\ , \tag{3.20}
$$

where S_F^{add} reduces to[7]

$$
S_F^{\mathrm{add}} = \frac{1}{2}\frac{\delta S_F}{\delta\phi^A}Y^A\lambda^2 - \frac{i\hbar}{2}(-1)^{\varepsilon_A}\,\frac{\delta Y^A}{\delta\phi^A}\lambda^2 \not\equiv 0\ , \tag{3.21}
$$

which implies that linearized finite BRST-antiBRST transformations can be interpreted neither as global symmetry transformations of the integrand nor as field-dependent transformations inducing an exact change of the gauge-fixing functional. Therefore, they do not possess the properties of finite BRST-antiBRST transformations.

[7]Even though in Yang–Mills theories there hold the properties $Y^i_{,i} = Y^\alpha_{,\alpha} = Y^{\alpha a}_{,\alpha a} = 0 \Longrightarrow (-1)^{\varepsilon_A}Y^A_{,A} = 0$, the quantity S_F^{add} does not vanish identically, $S_F^{\mathrm{add}} \not\equiv 0$, so that the invariance of the integrand in the vacuum functional under global linearized finite BRST-antiBRST transformations can only take place on solutions of the equation $S_{F,A}Y^A = 0$.

3.2 Constrained Dynamical Systems

The case of arbitrary dynamical systems with first-class constraints can be examined in complete analogy with the Yang–Mills case and is based on the propositions and considerations of Subsection 3.1. Namely, in the case of dynamical systems in question, the linear part of finite field-dependent BRST-antiBRST transformations for phase-space trajectories Γ_t^p has the form

$$\Gamma_t^p \to \check{\Gamma}_t^p = \Gamma_t^p + \Delta\Gamma_t^p \ , \quad \text{where} \quad \Delta\Gamma_t^p = \left(s^a\Gamma_t^p\right)\lambda_a = X_t^{pa}\lambda_a \ , \quad \lambda_a\left(\Gamma\right) = \int dt \ s_a\Lambda\left(\Gamma\right) \ . \tag{3.22}$$

Let us examine the even matrix $M = \|M^p_{q|t',t''}\|$

$$\frac{\delta\left(\Delta\Gamma^p_{t'}\right)}{\delta\Gamma^q_{t''}} = M^p_{q|t',t''} \ , \quad \varepsilon(M^p_{q|t',t''}) = \varepsilon_p + \varepsilon_q$$

and the corresponding Jacobian $\exp\left(\Im\right)$

$$\begin{aligned} &\Im = \operatorname{Str}\ln\left(\mathbb{I} + M\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\operatorname{Str}\left(M^n\right), \quad \operatorname{Str}\left(M^n\right) = (-1)^{\varepsilon_p}\int dt \ \left(M^n\right)^p_p(t,t) \ , \\ &\mathbb{I} = \delta^p_q\delta\left(t'-t''\right) \ , \quad (M)^p_q\left(t',t''\right) = \frac{\delta\Delta\Gamma^p\left(t'\right)}{\delta\Gamma^q\left(t''\right)} \ , \quad (AB)^p_q\left(t',t''\right) = \int dt \ (A)^p_r\left(t',t\right)B^r_q\left(t,t''\right) \ . \end{aligned} \tag{3.23}$$

Explicitly, the matrix M is given by the sum of two even matrices:

$$\begin{aligned} &M^p_{q|t',t''} = X^{pa}_{t'}\frac{\delta\lambda_a}{\delta\Gamma^q_{t''}} + \frac{\delta X^{pa}_{t'}}{\delta\Gamma^q_{t''}}\lambda_a\left(-1\right)^{\varepsilon_p} \equiv U^p_{q|t',t''} + V^p_{q|t',t''} \ , \\ &U^p_{q|t',t''} = X^{pa}_{t'}\frac{\delta\lambda_a}{\delta\Gamma^q_{t''}} \ , \quad (V)^p_{q|t',t''} = \frac{\delta X^{pa}_{t'}}{\delta\Gamma^q_{t''}}\lambda_a\left(-1\right)^{\varepsilon_q} \ . \end{aligned} \tag{3.24}$$

The matrices U, V correspond to the matrices P, Q of Subsection 3.1. This correspondence is given explicitly by Table 1.

First-class constraint systems	Yang–Mills theories
$\Gamma^p_t \ , \ \Delta\Gamma^p_t = \left(s^a\Gamma^p_t\right)\lambda_a \ , \ \lambda^a\left(\Gamma\right) = \int dt \ s^a\Lambda\left(\Gamma\right)$	$\phi^A \ , \ \Delta\phi^A = \left(s^a\phi^A\right)\lambda_a, \ \lambda^a\left(\phi\right) = s^a\Lambda\left(\phi\right)$
$s^a\Gamma^p_t = X^{pa}_t, \ s^bs^a\Gamma^p_t = \int dt' \ \frac{\delta X^{pa}_t}{\delta\Gamma^q_{t'}}X^{qb}_{t'} = \varepsilon^{ab}Y^p_t$	$s^a\phi^A = X^{Aa}, \ s^bs^a\phi^A = \frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb} = \varepsilon^{ab}Y^A$
$\int dt \ \frac{\delta X^{pa}_t}{\delta\Gamma^p_t} = 0, \ Y^p_t = -\frac{1}{2}\varepsilon_{ab}\int dt' \frac{\delta X^{pa}_t}{\delta\Gamma^q_{t'}}X^{Bb}_{t'}$	$\frac{\delta X^{Aa}}{\delta\phi^A} = 0, \ Y^A = -\frac{1}{2}\varepsilon_{ab}\frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb}$
$\frac{\delta\left(\Delta\Gamma^p_{t'}\right)}{\delta\Gamma^q_{t''}} = M^p_{q\|t',t''} = U^p_{q\|t',t''} + V^p_{q\|t',t''}$	$\frac{\delta\left(\Delta\phi^A\right)}{\delta\phi^B} = M^A_B = P^A_B + Q^A_B$
$U^p_{q\|t',t''} = X^{pa}_{t'}\frac{\delta\lambda_a}{\delta\Gamma^q_{t''}} \ , \ (V)^p_{q\|t',t''} = \frac{\delta X^{pa}_{t'}}{\delta\Gamma^q_{t''}}\lambda_a\left(-1\right)^{\varepsilon_q}$	$P^A_B = X^{Aa}\frac{\delta\lambda_a}{\delta\phi^B} \ , \ (Q)^A_B = \frac{\delta X^{Aa}}{\delta\phi^B}\lambda_a\left(-1\right)^{\varepsilon_B}$

Table 1: Correspondence of the matrix elements in arbitrary first-class constraint systems and Yang–Mills theories. Linearized field-dependent BRST-antiBRST transformations.

In this connection, due to the property $\operatorname{Str}(AB) = \operatorname{Str}(BA)$ for even matrices, Propositions 1, 2 of Subsection 3.1 remain formally the same[8] in terms of $U^p_{q|t',t''}$, $V^p_{q|t',t''}$, substituted instead of the respective matrices P^A_B, Q^A_B, which establishes the following

[8]One should, of course, take into account that formal summation over the time variable included in the index A is replaced by explicit integration over t in terms of (p,t).

Proposition 3 *The matrices (3.2) with arbitrary odd-valued $\lambda_a \not\equiv s_a \int dt\ \Lambda$ obey the properties*

$$\mathrm{Str}\,(U+V)^n = \mathrm{Str}\left(U^n + nU^{n-1}V + C_n^2 U^{n-2}V^2\right)\ , \quad \text{where} \quad C_n^k = \frac{n!}{k!\,(n-k)!}\ , \quad n=2,3\ , \tag{3.25}$$

$$\mathrm{Str}\,(V) = 0\ , \quad \mathrm{Str}\left(V^2\right) = 2\mathrm{Str}\,(W)\ , \quad \text{where} \quad W_q^p \equiv -\frac{1}{2}\lambda^2 \frac{\delta Y^p}{\delta \Gamma^q}\ . \tag{3.26}$$

Proposition 4 *Let us suppose that the condition $\lambda_a = \int dt\ s_a \Lambda$ is fulfilled. Then there hold the properties*[9]

$$U^2 = f\cdot U \Longrightarrow U^n = f\cdot U^{n-1}\ ,\ \text{where}\ s^a\lambda_b = \delta_b^a f\ \Longrightarrow f = -\frac{1}{2}\int dt\ s^2\Lambda = -\frac{1}{2}\mathrm{Str}\,(U)\ , \tag{3.27}$$

$$VU = (1+f)\cdot V_2\ ,\ \text{where}\ (V_2)_q^p \equiv \lambda_a Y^p \frac{\delta\lambda^a}{\delta\Gamma^q}(-1)^{\varepsilon_p+1}\ , \tag{3.28}$$

$$\mathrm{Str}\,(U+V)^n = \mathrm{Str}\left(U^n + nU^{n-1}V^1 + nU^{n-2}V^2 + K_n U^{n-3}VUV\right),\ \text{where}\ K_n \equiv C_n^2 - C_n^1 = \frac{n\,(n-3)}{2}\ ,\ n\geq 4\ . \tag{3.29}$$

Note: these statements may be supplied by the same remarks that follow Propositions 1, 2, with the replacement of ϕ^A, P_B^A, Q_B^A, R_B^A by Γ^p, $U^p_{q|t',t''}$, $V^p_{q|t',t''}$, $W^p_{q|t',t''}$, respectively, and with the replacement of $\lambda^a(\phi) = s^a\Lambda(\phi)$ by $\lambda^a(\Gamma) = \int dt\ s^a\Lambda(\Gamma)$; in particular, it may be emphasized that the properties (3.26), (3.27), (3.28) are implied by the invariance, $d\check{\Gamma} = d\Gamma$, of functional integration measure under global BRST-antiBRST transformations $\delta\Gamma^p = (s^a\Gamma^p)\lambda_a$, $\lambda_a = \text{const}$, being canonical transformations of phase-space variables, as well as by the anticommutativity, $s^a s^b + s^b s^a = 0$, of the corresponding generators s^a.

From (3.25)–(3.29), with allowance for $\mathrm{Str}\,(AB) = \mathrm{Str}\,(BA)$, it follows that $\Im$ acquires the form, cf. (3.30),

$$\Im = -2\ln(1+f) + \Re\ , \quad \Re = -\mathrm{Str}\left[W + V_2 + (1/2)\,V_2^2 - (1+f)\,VV_2\right]\ , \tag{3.30}$$

where

$$f = -\frac{1}{2}\int dt\ s^2\Lambda = \frac{1}{2}\int dt\ s^a s_a\Lambda\ .$$

Substituting the explicit form of the matrices (3.24), (3.26), (3.28), we have, cf. (3.10),

$$\Re = \frac{1}{2}(-1)^{\varepsilon_p}\int dt\left[\frac{\delta\left(Y_t^p\lambda^2\right)}{\delta\Gamma_t^p} - \frac{1}{4}Y_t^p\frac{\delta\lambda^2}{\delta\Gamma_t^q}Y_t^q\frac{\delta\lambda^2}{\delta\Gamma_t^p} - (1+f)\frac{\delta X_t^{pa}}{\delta\Gamma_t^q}Y_t^q\lambda_a\frac{\delta\lambda^2}{\delta\Gamma_t^p}\right]\ . \tag{3.31}$$

Using the relations $X^{pa} = s^a\Gamma^p$, $Y^p = -(1/2)\,s^2\Gamma^p$ and bearing in mind that $\lambda_a = \int dt\ s_a\Lambda$, one can represent the contribution (3.31) in terms of BRST-antiBRST variations, cf. (3.11),

$$\begin{aligned}\Re = &-\frac{1}{4}(-1)^{\varepsilon_p}\int dt\left[\left(s^2\Gamma_t^p\right)\lambda^2\right]_{,(p,t)} - \frac{1}{32}(-1)^{\varepsilon_p}\int dt'dt''\left(s^2\Gamma_{t'}^p\right)\lambda^2_{,(q,t'')}\left(s^2\Gamma_{t''}^q\right)\lambda^2_{,(p,t')}\\ &+\frac{1}{4}(-1)^{\varepsilon_p}\left(1-\frac{1}{2}\int dt\ s^2\Lambda\right)\int dt'dt''\left(s^a\Gamma_{t'}^p\right)_{,(q,t'')}\left(s^2\Gamma_{t''}^q\right)\lambda_a\lambda^2_{,(p,t')}\ ,\end{aligned} \tag{3.32}$$

where

$$A_{,(p,t)} \equiv \frac{\delta A}{\delta\Gamma_t^p} = A\frac{\overleftarrow{\delta}}{\delta\Gamma_t^p}\ .$$

By analogy with Subsection 3.1, one can state that the linearized finite BRST-antiBRST transformations (3.22) for dynamical systems with first-class constraints can be interpreted neither as global symmetry transformations of the integrand, nor as field-dependent transformations inducing an exact change of the gauge-fixing functional. Therefore, linearized finite BRST-antiBRST transformations in Hamiltonian formalism do not possess the properties of finite BRST-antiBRST transformations.

[9]Further on, we will use different forms of the same matrices: $V_q^p = X_{,q}^{pa}\lambda_a(-1)^{\varepsilon_q} = \lambda_a X_{,q}^{pa}(-1)^{\varepsilon_p+1}$, $(V_2)_q^p = \lambda_a Y^p\lambda^a_{,q}(-1)^{\varepsilon_p+1} = -(1/2)\,Y^p\lambda^2_{,p}$.

4 Finite BRST-antiBRST Transformations with Arbitrary Parameters

In this section, we examine the calculation of the Jacobian for finite field-dependent BRST-antiBRST transformations in the case of arbitrary, i.e., generally independent parameters, $\lambda_a \not\equiv s_a\Lambda$. Once again, we shall carry out the explicit calculations in the Yang–Mills case and then make a relation of the resulting Jacobian to the case of arbitrary dynamical systems with first-class constraints. Furthermore, as long as the case of general gauge theories in Lagrangian formalism proves similar to the Yang–Mills case, the corresponding general considerations will be provided as well. The calculations in Yang–Mills theories and first-class constraint systems will partially repeat the case of linearized BRST-antiBRST transformations and will therefore effectively use some of the corresponding statements given by the above propositions. At the same time, we will slightly change the notation (3.2), (3.24) of the matrix objects for the sake of convenience.

4.1 Yang–Mills Theories

In the Yang–Mills case, the finite field-dependent BRST-antiBRST transformations in question have the form

$$\phi^A \to \phi'^A = \phi^A + \Delta\phi^A \ , \quad \text{where} \quad \Delta\phi^A = (s^a\phi)\,\lambda_a + \frac{1}{4}\left(s^2\phi\right)\lambda^2 = X^{Aa}\lambda_a - \frac{1}{2}Y^A\lambda^2 \ , \quad \lambda_a \not\equiv s_a\Lambda \ .$$

Let us examine the corresponding even matrix $M = \|M^A_B\|$ and the related quantity $\Im$

$$M^A_B = \frac{\delta\left(\Delta\phi^A\right)}{\delta\phi^B} \ , \quad \Im = \mathrm{Str}\ln\left(\mathbb{I}+M\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(M^n\right) \ .$$

Explicitly, the matrix M^A_B is given by the sum of three even matrices:

$$M^A_B = P^A_B + Q^A_B + R^A_B\,, \ \text{where} \ Q^A_B = (Q_1)^A_B + (Q_2)^A_B \ , \tag{4.1}$$

$$P^A_B = X^{Aa}\frac{\delta\lambda_a}{\delta\phi^B} \ , \ (Q_1)^A_B = \lambda_a\frac{\delta X^{Aa}}{\delta\phi^B}(-1)^{\varepsilon_A+1} \ , \ (Q_2)^A_B = \lambda_a Y^A\frac{\delta\lambda^a}{\delta\phi^B}(-1)^{\varepsilon_A+1} \ , \ R^A_B = -\frac{1}{2}\lambda^2\frac{\delta Y^A}{\delta\phi^B} \ . \tag{4.2}$$

Here, the matrix Q^A_B of Subsection 3.1 has been naturally extended by its summation with the matrix $(Q_2)^A_B$, which has already emerged in the relation (3.7) of the mentioned subsection. The additional matrix R^A_B has also emerged (3.5) in Subsection 3.1.

Using the property $\mathrm{Str}\,(AB) = \mathrm{Str}\,(BA)$ for arbitrary even matrices and the fact that the occurrence of $R \sim \lambda^2$ in $\mathrm{Str}\,(M^n)$ more than once yields zero, $\lambda^4 \equiv 0$, we have

$$\mathrm{Str}\left(M^n\right) = \mathrm{Str}\left(P+Q+R\right)^n = \sum_{k=0}^{1} C^k_n \mathrm{Str}\left[\left(P+Q\right)^{n-k}R^k\right] \ , \quad C^k_n = \frac{n!}{k!\,(n-k)!} \ . \tag{4.3}$$

Moreover,

$$\mathrm{Str}\left(P+Q+R\right)^n = \mathrm{Str}\left(P+Q\right)^n + n\mathrm{Str}\left[\left(P+Q\right)^{n-1}R\right] = \mathrm{Str}\left(P+Q\right)^n + n\mathrm{Str}\left(P^{n-1}R\right) \ , \tag{4.4}$$

since any occurrence of $R \sim \lambda^2$ and $Q \sim \lambda_a$ simultaneously entering $\mathrm{Str}\,(M)^n$ yields zero, owing to $\lambda_a\lambda^2 = 0$, as a consequence of which R can only be coupled with P^{n-1}.

Further considerations are based on the following statements, proved in Appendices B.1– B.5, respectively:

Lemma 1 *The expressions* $\mathrm{Str}\,(M^n)$ *for* $n \geq 1$ *are given by*

$$\mathrm{Str}\left(M^n\right) = \mathrm{Str}\left(P+Q\right)^n + n\mathrm{Str}\left(P^{n-1}R\right) = \begin{cases} \mathrm{Str}\left(P+Q\right) + \mathrm{Str}\left(R\right) \ , & n = 1 \ , \\ \mathrm{Str}\left(P+Q\right)^n \ , & n > 1 \ . \end{cases} \tag{4.5}$$

Note: the relation (4.5) uses the nilpotency $s^a s^b s^c \equiv 0$ of the generators s^a, as a consequence of their anticommutativity, and implies that the matrix R drops out of $\mathrm{Str}\,(M^n)$, $n>1$, and enters the quantity $\Im$ only as $\mathrm{Str}\,(R)$.

Lemma 2 *The expressions* $\mathrm{Str}\,(P+Q)^n$ *for* $n>1$ *are given by*

$$\mathrm{Str}\,(P+Q)^n = \sum_{k=0}^{n} C_n^k \mathrm{Str}\left(P^{n-k}Q^k\right) = \mathrm{Str}\left(P^n + C_n^1 P^{n-1}Q + C_n^2 P^{n-2}Q^2\right) \ , \quad n=2,3 \ , \tag{4.6}$$

$$\mathrm{Str}\,(P+Q)^{2k} = \sum_{l=0}^{1} C_{2k}^l \mathrm{Str}\left(P^{2k-l}Q^l\right) + C_{2k}^1 \sum_{l=0}^{k-2} \mathrm{Str}\left[P^{2(k-l-1)}\left(P^lQ\right)^2\right] + C_k^1 \mathrm{Str}\left[\left(P^{k-1}Q\right)^2\right], \quad k\geq 2 \ , \tag{4.7}$$

$$\mathrm{Str}\,(P+Q)^{2k+1} = \sum_{l=0}^{1} C_{2k+1}^l \mathrm{Str}\left(P^{2k+1-l}Q^l\right) + C_{2k+1}^1 \sum_{l=0}^{k-1} \mathrm{Str}\left[P^{2(k-l)-1}\left(P^lQ\right)^2\right] \ , \quad k\geq 2 \ . \tag{4.8}$$

Note: the relation (4.6) coincides with the formula (3.4) of Proposition (1), whereas the relations (4.7), (4.8) generalize the formula (3.8) of Proposition (2) to the case $\lambda_a \not\equiv s_a\Lambda$. Indeed, let us suppose that the case $\lambda_a = s_a\Lambda$, with the implied condition $P^n = f\cdot P^{n-1}$, does indeed take place. Then it is straightforward to verify the equalities

$$\left.\begin{array}{l}\mathrm{Str}\,(P+Q)^{2k}\\ \mathrm{Str}\,(P+Q)^{2k+1}\end{array}\right\} = \mathrm{Str}\,(P^n) + n\mathrm{Str}\left(P^{n-1}Q\right) + n\mathrm{Str}\left(P^{n-2}Q^2\right) + K_n\mathrm{Str}\left(P^{n-3}QPQ\right) \ , \quad k\geq 2 \ , \tag{4.9}$$

where the coefficients K_n are given by (3.8), with allowance for

$$K_{2k} = (k-2)\,C_{2k}^1 + C_k^1 \ , \quad K_{2k+1} = (k-1)\,C_{2k+1}^1 \ , \tag{4.10}$$

which shows that in the respective cases $n=(2k,2k+1)$ the above relations for $\mathrm{Str}\,(P+Q)^{2k}$ and $\mathrm{Str}\,(P+Q)^{2k+1}$ are reduced to the formula (3.8) for $\mathrm{Str}\,(P+Q)^n$, when $\lambda_a = s_a\Lambda$.

Lemma 3 *There hold the properties*

$$\mathrm{Str}\,(Q_1) = 0 \ , \quad \mathrm{Str}\,(R) - \frac{1}{2}\mathrm{Str}\left(Q_1^2\right) = 0 \ . \tag{4.11}$$

Note: the relations (4.11) repeat, in different notation, the formulas (3.5) of Proposition 1, established in our paper [1]; for the sake of completeness of the present subsection, we will provide the corresponding proof in Appendix B.3.

Lemma 4 *There hold the properties*

$$\mathrm{Str}\,(P^n) = -\mathrm{tr}\,[(m^n)^a_b] \equiv -\mathrm{tr}\,(m^n) = -\,(m^n)^a_a \ , \quad \text{where} \quad m^a_b \equiv s^a\lambda_b \ , \tag{4.12}$$

where powers in $m=m^a_b$ *are understood in the sense of matrix multiplication with respect to* $\mathrm{Sp}(2)$ *indices.*

Lemma 5 *There hold the properties*

$$QP^n = \mathrm{tr}\left[m^{n-1}\,(e+m)\,Y\right] \ , \quad n\geq 1 \ , \tag{4.13}$$

where $e=(e)^a_b$ *is the unit matrix* $(e)^a_b\equiv\delta^a_b$*, and the matrix* $Y=(Y^a_b)^A_B$ *is given by*

$$(Y^a_b)^A_B \equiv (-1)^{\varepsilon_A}\,\lambda^a Y^A \frac{\delta\lambda_b}{\delta\phi^B} \Longrightarrow (Y^a_a)^A_B = (Q_2)^A_B \ . \tag{4.14}$$

Note: the relations (4.12), (4.13) generalize the respective formulae (3.6), (3.7) of Proposition 2 to the case $\lambda_a \not\equiv s_a\Lambda$, which is readily established by inserting the particular form of the matrix $m^a_b \sim \delta^a_b$,

$$m^a_b = s^a\lambda_b = s^a s_b\Lambda = -\frac{1}{2}\delta^a_b\left(s^2\Lambda\right) \ , \tag{4.15}$$

corresponding to the case $\lambda_a = s_a\Lambda$, in the relations (4.12), (4.13), with the resulting formulae (3.6), (3.7); due to the natural appearance of the matrices m^a_b and $(Y^a_b)^A_B$ in (4.12), (4.13), we shall evaluate the quantity $\Im$ as a series in powers of these objects.

Proceeding to the calculation of $\Im$ on the basis of the above lemmas and collecting the relations (4.3)–(4.8), (4.11)–(4.14), we arrive (see Appendix B.6) at the following result:

$$\Im = -\mathrm{tr}\ln\left(e+m\right) \ , \tag{4.16}$$

where the operation ln is to be understood in the sense of an expansion in powers with respect to the multiplication of matrices carrying Sp(2) indices:

$$\ln\left(e+m\right) = \left[\ln\left(e+m\right)\right]^a_b = -\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}\left(m^n\right)^a_b \ . \tag{4.17}$$

It should be emphasized that the considerations of Appendix B.6 do not utilize the anticommutativity of the BRST-antiBRST generators s_a, except for the treatment of $\mathrm{Str}\left(P^{n-1}R\right)$, $\mathrm{Str}\left(R\right)$, $\mathrm{Str}\left(Q_1^2\right)$ in (4.5) and (4.11). In the remaining part of this subsection, we examine the consequences implied in (4.16), (4.17) by the anticommutativity of s_a. Namely, in the particular case $\lambda_a = s_a\Lambda$, the quantity $\Im$ reduces, in accordance with (4.15),

$$\left(m^n\right)^a_b = f^n\cdot\delta^a_b \ , \quad \mathrm{tr}\left(m^n\right) = 2f^n \ , \quad f = -\frac{1}{2}s^2\Lambda \ , \tag{4.18}$$

to the BRST-antiBRST-exact expression [1]

$$\Im = \sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}\mathrm{tr}\left(m^n\right) = 2\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}f^n = -2\ln\left(1+f\right) = \ln\left(1-\frac{1}{2}s^2\Lambda\right)^{-2} \ . \tag{4.19}$$

In the general case, however, $\lambda_a \not\equiv s_a\Lambda$, the quantity $\Im$ fails to be BRST-antiBRST-invariant,

$$\begin{aligned} s^a\Im &= \sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}s^a\left(m^n\right)^b_b = \sum_{n=1}^{\infty}\left(-1\right)^n\left(s^a m^b_c\right)\left(m^{n-1}\right)^c_b \\ &= -\left(s^a m^b_c\right)\sum_{k=0}^{\infty}\left(-1\right)^k\left(m^k\right)^c_b = -\left(s^a m^b_c\right)\left[\left(e+m\right)^{-1}\right]^c_b \not\equiv 0 \ , \end{aligned} \tag{4.20}$$

whence it is generally no longer BRST-antiBRST-exact and does not amount to an exact change of the gauge-fixing Boson:

$$\Im \not\equiv \frac{1}{2i\hbar}s^2\Delta F \ . \tag{4.21}$$

The condition of BRST-antiBRST-invariance of $\Im$ therefore reads

$$\left(s^a m^b_c\right)\left[\left(e+m\right)^{-1}\right]^c_b = \frac{1}{2}\varepsilon^{ab}\left(s^2\lambda_c\right)\left[\left(e+m\right)^{-1}\right]^c_b = 0 \ , \tag{4.22}$$

which is a necessary condition of BRST-antiBRST-exactness of $\Im$. Furthermore, if we impose on $\Im\left(\lambda\right)$, given by an expansion in powers of λ_a,

$$\Im\left(\lambda\right) = -s^a\lambda_a + \frac{1}{2}\left(s^a\lambda_b\right)\left(s^b\lambda_a\right) - \frac{1}{3}\left(s^a\lambda_b\right)\left(s^b\lambda_c\right)\left(s^c\lambda_a\right) + \cdots \ , \tag{4.23}$$

the requirement of BRST-antiBRST-exactness at the first order, $s^a\lambda_a = s^a s_a\Lambda$, for a certain even-valued functional Λ, then this requirement meets the condition (4.22) and turns out to provide the corresponding exactness at the succeeding orders, which implies the following (see Appendix B.7)

Lemma 6 *If there exists an even-valued functional* Λ *such that* $s^a\lambda_a = s^a s_a\Lambda$*, then there also exists a sequence of* Λ_n *such that*

$$\mathrm{tr}\left(m^n\right) = s^a s_a \Lambda_n \ , \quad n \geq 2 \implies \Im = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} s^a s_a \Lambda_n \ . \tag{4.24}$$

This implies the following criterion: the quantity $\Im(\lambda)$ is BRST-antiBRST-exact to all orders of its expansion in powers of λ_a if and only if there exists such an even-valued functional Λ that $s^a\lambda_a = -s^2\Lambda$. Such a choice of λ_a obviously corresponds to the case of functionally-dependent parameters,

$$s^1\lambda_1 + s^2\lambda_2 = -s^2\Lambda \ , \tag{4.25}$$

which we have previously examined [1] in the particular case $\lambda_a = s_a\Lambda$. Since an arbitrary set of functional parameters $\lambda_a(\phi)$ is generally not functionally-dependent, $s^a\lambda_a \not\equiv -s^2\Lambda$, it is obvious that the corresponding quantum action induced by a finite BRST-antiBRST transformation with such parameters $\lambda_a(\phi)$ *cannot be reproduced* by the conventional Lagrangian BRST-antiBRST quantization scheme.

It has been previously established [1] that the particular case $\lambda_a = s_a\Lambda$ of functionally-dependent parameters λ_a allows one to obtain a unique solution of the corresponding compensation equation

$$\ln\left(1 - \frac{1}{2}s^2\Lambda\right)^{-2} = \frac{1}{2i\hbar}s^2\Delta F \ ,$$

with accuracy up to BRST-antiBRST-exact terms. This is a consequence of the fact that the resulting quantity $\Im$ is actually controlled by a single functional parameter Λ, which is in one-to-one correspondence (up to the above-mentioned accuracy) with a change ΔF of the gauge Boson. In this respect, it is natural to examine the most general case of solutions to $s^a\lambda_a = s^a s_a\Lambda$, parameterized by an additional odd-valued doublet ψ_a,

$$s^a\left(\lambda_a - s_a\Lambda\right) = 0 \Longrightarrow \lambda_a = s_a\Lambda + \psi_a \ , \quad s^a\psi_a = 0 \ , \tag{4.26}$$

which may, in particular, be constant, $\psi_a = \mathrm{const}$. In virtue of (4.26), the additional parameters ψ_a are functionally-dependent and obey (see Appendix B.8)

Lemma 7 *The condition* $s^a\psi_a = 0$ *implies*

$$\mathrm{tr}\left(m_\psi^2\right) = -\frac{1}{2}s^2\left(\psi^2\right) \ , \quad \mathrm{tr}\left(m_\psi^n\right) = 0 \ , \quad \text{for} \quad n \geq 3 \ , \quad \text{where} \quad \psi^2 \equiv \psi_a\psi^a \ , \quad \left(m_\psi\right)^a_b \equiv s^a\psi_b \ , \tag{4.27}$$

whence the corresponding quantity $\Im$*, parameterized by the functional parameters* (Λ, ψ_a)*, is BRST-antiBRST-exact and reads as follows:*

$$\Im(\Lambda, \psi) = \ln\left(1 - \frac{1}{2}s^2\Lambda\right)^{-2} - \frac{1}{4}s^2\left(\psi^2\right)\left[\left(1 - \frac{1}{2}s^2\Lambda\right)^{-2}\right] \ . \tag{4.28}$$

As a consequence, the resulting compensation equation takes the form

$$\ln\left(1 - \frac{1}{2}s^2\Lambda\right)^{-2} - \frac{1}{4}s^2\left(\psi^2\right)\left[\left(1 - \frac{1}{2}s^2\Lambda\right)^{-2}\right] = \frac{1}{2i\hbar}s^2\Delta F \ . \tag{4.29}$$

For a given gauge variation ΔF and a certain given solution ψ_a of the subsidiary condition (4.26), the modified compensation equation (4.29) may be considered as an equation for some unknown functional Λ, whose solution may be sought as $\Lambda = \Lambda(\Delta F, \psi)$. More explicitly, there holds (see Appendix B.9) the following

Lemma 8 *The solutions Λ of the modified compensation equation (4.29) have the form*

$$s^2\Delta F \neq 0: \quad \Lambda\left(\Delta F, \psi\right) = \frac{2\Delta F}{s^2\Delta F}\left\{1 - \left[1+\vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}} X_0^{-\frac{1}{2}}\right\}\Big|_{X_0=\exp\left(\frac{1}{2i\hbar}s^2\Delta F\right), \gamma=\frac{1}{4}s^2\left(\psi^2\right)} \ , \tag{4.30}$$

$$s^2\Delta F = 0 \ , \ s^2\left(\psi^2\right) = 0: \quad \Lambda = s^a\tilde{\lambda}_a + s^2\tilde{\Lambda} \ , \tag{4.31}$$

$$s^2\Delta F = 0, \ s^2\left(\psi^2\right) \neq 0: \quad \Lambda\left(\psi\right) = \frac{2\psi^2}{s^2\left(\psi^2\right)}\left\{1 - \left[1+\vartheta\left(\gamma\right)\right]^{-\frac{1}{2}}\right\}\Big|_{\gamma=\frac{1}{4}s^2\left(\psi^2\right)} \ , \tag{4.32}$$

where the function $\vartheta\left(y\right)$ is defined by

$$\theta\left(x\right) = \frac{\ln\left(1+x\right)}{\left(1+x\right)} \ , \quad \theta\left(0\right) = 0 \ ,$$

$$\vartheta\left(y\right): \vartheta\left(\theta\left(x\right)\right) = x \ , \quad \vartheta\left(0\right) = 0 \ .$$

Therefore, in the case $s^2\Delta F \neq 0$, $s^2\left(\psi^2\right) = 0$, we find the solution (2.22),

$$\Lambda\left(\Delta F, 0\right) = \frac{2\Delta F}{s^2\Delta F}\left\{1 - \exp\left[\left(i/4\hbar\right)s^2\Delta F\right]\right\} \ ,$$

of the usual compensation equation (2.20), whereas in the case $s^2\Delta F = 0$, $s^2\left(\psi^2\right) \neq 0$ we arrive at a finite BRST-antiBRST transformation, with the parameters $\lambda_a = s_a\Lambda\left(\psi\right) + \psi_a$ given by (4.26), (4.32), which induces a Jacobian equal to unity:

$$\Im = \frac{1}{2i\hbar}s^2\Delta F = 0 \Longrightarrow \exp\left(\Im\right) = 1 \ . \tag{4.33}$$

On the other hand, given the functionals Λ and ψ_a, we obtain a change of the gauge ΔF, according to (4.29), which, in the case $\Lambda = 0$, takes the form

$$s^2\Delta F = -\frac{i\hbar}{2}s^2\left(\psi^2\right) \tag{4.34}$$

and corresponds to the quantity $\Im\left(\psi\right)$ given by

$$\Im\left(\psi\right) = -\frac{1}{4}s^2\left(\psi^2\right) \ , \tag{4.35}$$

which implies a non-trivial Jacobian, $\exp\left(\Im\right) \neq 1$, in the case $s^2\left(\psi^2\right) \neq 0$. In order to investigate this possibility in more detail, let us notice that the solutions of the subsidiary condition (4.26) can be presented in the form

$$\psi_a = \mu_a + \frac{1}{2}s_a\Psi^b_b + s^b\Psi_{ba} + s_bs^b\varphi_a \ , \quad \Psi^a_b \equiv \varepsilon^{ac}\Psi_{cb} \ ,$$
$$\varepsilon(\mu_a) = \varepsilon(\varphi_a) = 1 \ , \quad \varepsilon(\Psi_{ab}) = 0 \ , \tag{4.36}$$

parameterized by a constant Sp(2)-doublet, $\mu_a = \text{const}$, an Sp(2)-doublet of arbitrary functionals, $\varphi_a\left(\phi\right)$, and an Sp(2)-tensor of arbitrary functionals, $\Psi_{ab}\left(\phi\right)$, with the corresponding Grassmann parities (4.36). The above solutions can be found from the following Ansatz:

$$\psi_a = \mu_a + s_a\Psi + s^b\Psi_{ba} + s_bs^b\varphi_a \ , \tag{4.37}$$

expanding the functionals ψ_a in powers of the operators s_a. Once a certain solution (4.36) is given, one can decompose the corresponding tensor Ψ_{ab} into its symmetric and antisymmetric components,

$$\Psi_{ab} = \Psi_{\{ab\}} + \Psi_{[ab]} \ ,$$

and notice that the antisymmetric component, $\Psi_{[ab]} \equiv \varepsilon_{ab}\Psi$, $\Psi = (1/2)\,\varepsilon^{ba}\Psi_{[ab]}$, actually vanishes from ψ_a:

$$\frac{1}{2}\varepsilon^{bc}s_a\Psi_{[cb]} + s^b\Psi_{[ba]} = \frac{1}{2}\varepsilon^{bc}\varepsilon_{cb}s_a\Psi + \varepsilon_{ba}s^b\Psi = s_a\Psi - s_a\Psi \equiv 0 \ .$$

Therefore, regular solutions of the equation $s^a\psi_a = 0$ in (4.26) vanishing in the case $\phi^A = 0$ have the form

$$\psi_a = s^b \Psi_{\{ba\}} + s^2 \varphi_a \ , \tag{4.38}$$

which is a particular case ($n = 1$) of a regular solution, vanishing in the case $\phi^A = 0$, of a more general equation for an unknown completely symmetric Sp(2)-tensor of rank n,

$$s^{a_1} \psi_{\{a_1 a_2 \ldots a_n\}} = 0 \Longleftrightarrow \psi_{\{a_1 a_2 \ldots a_n\}} = s^b \Psi_{\{b a_1 a_2 \ldots a_n\}} + s^2 \varphi_{\{a_1 a_2 \ldots a_n\}} \ , \tag{4.39}$$

with certain rank-n and rank-$(n+1)$ symmetric Sp(2)-tensors $\varphi_{\{a_1 a_2 \ldots a_n\}}$, $\Psi_{\{b a_1 a_2 \ldots a_n\}}$, $\varepsilon(\varphi) = \varepsilon(\psi) = \varepsilon(\Psi)+1$. It can next be noticed that the components μ_a and φ_a in (4.37) do not contribute to $(m_\psi)^a_b = s^a\psi_b$, whereas the symmetric component $\Psi_{\{ab\}}$ (once non-vanishing) does,

$$(m_\psi)^a_b = s^a\psi_b = s^a \left(\frac{1}{2}\varepsilon^{cd} s_b \Psi_{\{dc\}} + s^c \Psi_{\{cb\}} \right) = \frac{1}{2}\varepsilon^{ac} s^2 \Psi_{\{cb\}} \not\equiv 0 \ ,$$

and furthermore it provides a non-vanishing contribution to $\mathrm{tr}(m_\psi) = (m_\psi)^a_b (m_\psi)^b_a$,

$$(m_\psi)^a_b \, (m_\psi)^b_a = \frac{1}{4}\varepsilon^{ac}\varepsilon^{bd} \left(s^2\Psi_{\{cb\}}\right) \left(s^2\Psi_{\{da\}}\right) = -\frac{1}{2} s^2 \left(\psi^2\right) \ ,$$

which makes it possible to express the quantity $\Im$ in (4.35) entirely in terms of the symmetric component:

$$\Im = \frac{1}{8}\varepsilon^{ac}\varepsilon^{bd} \left(s^2\Psi_{\{ad\}}\right) \left(s^2\Psi_{\{bc\}}\right) \ . \tag{4.40}$$

Finally, in the most general case of arbitrary functionals $\lambda_a(\phi)$, the condition (4.26) is not fulfilled, making it thereby impossible to present the quantity $\Im$ in a BRST-antiBRST-exact form (4.28) and to relate it with some change of the gauge (4.29). This means that the corresponding quantity $\Im$ acquires some extra contributions w.r.t. (4.28), which can be related to a decomposition of the parameters λ_a into the following components:

$$\lambda_a = s_a \Lambda + \psi_a + \sigma_a \ ,$$

where

$$s^a\psi_a = 0 \ , \quad s^a\sigma_a \neq 0 \ , \quad s^a\sigma_b \neq \delta^a_b f' \ .$$

Using the notation

$$(m_\Lambda)^a_b = s^a s_b \Lambda = \delta^a_b f \ , \quad (m_\psi)^a_b = s^a\psi_b \ , \quad (m_\sigma)^a_b = s^a\sigma_b \ ,$$

and considerations similar to the relations (B.78), (B.79), (B.80) of Appendix B.8, we have

$$\mathrm{tr}\left(m_\Lambda + m_\psi + m_\sigma\right)^n = \mathrm{tr}\sum_{k=0}^{n} C^k_n f^{n-k} \left(m_\psi + m_\sigma\right)^k \ , \tag{4.41}$$

whence the corresponding quantity $\Im = \Im(\Lambda, \psi, \sigma)$ reads

$$\Im(\Lambda,\psi,\sigma) = \ln\left(1 - \frac{1}{2}s^2\Lambda\right)^{-2} + M(\Lambda,\psi,\sigma) \ , \quad M(\Lambda,\psi,\sigma) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{k=1}^{n} C^k_n f^{n-k} \mathrm{tr}\left[\left(m_\psi + m_\sigma\right)^k\right]_{f=-\frac{1}{2}s^2\Lambda} \ . \tag{4.42}$$

Using the fact that $\mathrm{tr}(m_\psi) = 0$, we find

$$M(\Lambda,\psi,\sigma) = -\mathrm{tr}\left(m_\sigma\right)\left(1 + \frac{1}{2}s^2\Lambda\right) + \frac{1}{2}\mathrm{tr}\left(m_\psi + m_\sigma\right)^2 + \sum_{n=3}^{\infty} \frac{(-1)^n}{n} \sum_{k=1}^{n} C^k_n f^{n-k} \mathrm{tr}\left[\left(m_\psi + m_\sigma\right)^k\right]_{f=-\frac{1}{2}s^2\Lambda} \ , \tag{4.43}$$

where account is to be taken of

$$\operatorname{tr}\left(m_\psi^2\right) = -\frac{1}{2}s^2\left(\psi^2\right)\ , \quad \operatorname{tr}\left(m_\psi^k\right) \equiv 0\ , \quad k \geq 3\ . \tag{4.44}$$

Accordingly, the corresponding quantity $\Im\left(\Lambda, \psi, \sigma\right)$ is given by

$$\Im\left(\Lambda, \psi, \sigma\right) = \Im\left(\Lambda, \psi\right) + \Re\left(\Lambda, \psi, \sigma\right)\ , \tag{4.45}$$

where the quantity $M\left(\Lambda, \psi, \sigma\right)$, given by (4.43), has been decomposed as

$$M\left(\Lambda, \psi, \sigma\right) = M\left(\Lambda, \psi\right) + \Re\left(\Lambda, \psi, \sigma\right)\ , \quad M\left(\Lambda, \psi\right) \equiv \left. M\left(\Lambda, \psi, \sigma\right)\right|_{\sigma=0}\ . \tag{4.46}$$

In (4.45), $\Im\left(\Lambda, \psi\right)$ has the form (4.28) and thereby represents the BRST-antiBRST-exact contribution, whereas $\Re\left(\Lambda, \psi, \sigma\right)$ represents the contribution

$$\Re\left(\Lambda, \psi, \sigma\right) = M\left(\Lambda, \psi, \sigma\right) - M\left(\Lambda, \psi\right)\ , \tag{4.47}$$

which is not BRST-antiBRST-exact and cannot be, therefore, reproduced by the conventional BRST-antiBRST quantization scheme; instead, it should be regarded as an addition to the transformed quantum action in the integrand of (2.1)

$$\mathcal{I}^F_{\phi g(\lambda(\phi))} = d\phi\ \exp\left\{(i/\hbar)\left[S_0 + (1/2)\, s^a s_a\left(F + \Delta F\right) - i\hbar\Re\left(\Lambda, \psi, \sigma\right)\right]\right\}\ , \tag{4.48}$$

calculated in the reference frame with the gauge Boson $F + \Delta F(\Lambda, \psi)$.

4.2 Constrained Dynamical Systems

The case of arbitrary dynamical systems with first-class constraints can be examined in complete analogy with the case of Yang–Mills theories. It is based on the propositions and considerations of Subsection 4.1 and repeats, in part, the considerations of Subsection 3.2. Namely, in the case of dynamical systems in question, the finite field-dependent BRST-antiBRST transformations with arbitrary parameters have the form

$$\Gamma^p_t \to \check{\Gamma}^p_t = \Gamma^p_t + \Delta\Gamma^p_t\ , \quad \text{where} \quad \Delta\Gamma^p_t = \left(s^a\Gamma^p_t\right)\lambda_a + \frac{1}{4}\left(s^2\Gamma^p_t\right)\lambda^2 = X^{pa}_t\lambda_a - \frac{1}{2}Y^p_t\lambda^2\ , \quad \lambda_a \not\equiv \int dt\ s_a\Lambda\ .$$

Let us examine the corresponding even matrix $M = \|M^p_{q|t',t''}\|$ and the related quantity $\Im$

$$M^p_{q|t',t''} = \frac{\delta\left(\Delta\Gamma^p_{t'}\right)}{\delta\Gamma^q_{t''}}\ , \quad \Im = \operatorname{Str}\ln\left(\mathbb{I} + M\right) = -\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n}\operatorname{Str}\left(M^n\right)\ .$$

Explicitly, the matrix $M^p_{q|t',t''}$ is given by the sum of three even matrices:

$$M^p_{q|t',t''} = U^p_{q|t',t''} + V^p_{q|t',t''} + W^p_{q|t',t''}\ , \ \text{where}\ V^p_{q|t',t''} = \left(V_1\right)^p_{q|t',t''} + \left(V_2\right)^p_{q|t',t''}\ , \tag{4.49}$$

$$U^p_{q|t',t''} = X^{pa}_{t'}\frac{\delta\lambda_a}{\delta\Gamma^q_{t''}}\ ,\ \left(V_1\right)^p_{q|t',t''} = \lambda_a\frac{\delta X^{pa}_{t'}}{\delta\Gamma^q_{t''}}\left(-1\right)^{\varepsilon_p+1}\ ,\ \left(V_2\right)^p_{q|t',t''} = \lambda_a Y^p_{t'}\frac{\delta\lambda^a}{\delta\Gamma^q_{t''}}\left(-1\right)^{\varepsilon_p+1}\ ,\ W^p_{q|t',t''} = -\frac{1}{2}\lambda^2\frac{\delta Y^p_{t'}}{\delta\Gamma^q_{t''}}\ . \tag{4.50}$$

Here, the matrix $(V)^p_{q|t',t''}$ of Subsection 3.2 has been naturally extended by its summation with the matrix $(V_2)^p_{q|t',t''}$, which has already emerged in the relation (3.28) of the mentioned subsection. The additional matrix $W^p_{q|t',t''}$ has also emerged (3.26) in Subsection 3.2. The matrices U, V, W correspond to the matrices P, Q, R of Subsection 4.1. This correspondence is given explicitly by Table 2.

In this connection, due to the property $\operatorname{Str}\left(AB\right) = \operatorname{Str}\left(BA\right)$, Lemmas 1–5 of Subsection 4.1 remain formally the same (see Footnote 8) in terms of $U^p_{q|t',t''}$, $V^p_{q|t',t''}$, $W^p_{q|t',t''}$ substituted instead of the respective matrices P^A_B, Q^A_B, R^A_B, which establishes the following

First-class constraint systems	Yang–Mills theories
$\Gamma^p_t\ ,\ \Delta\Gamma^p_t=\left(s^a\Gamma^p_t\right)\lambda_a+\frac{1}{4}\left(s^2\Gamma^p_t\right)\lambda^2$	$\phi^A\ ,\ \Delta\phi^A=\left(s^a\phi^A\right)\lambda_a+\frac{1}{4}\left(s^2\phi^A\right)\lambda^2$
$s^a\Gamma^p_t=X^{pa}_t,\ s^bs^a\Gamma^p_t=\varepsilon^{ab}Y^p_t\ ,\ s^cs^bs^a\Gamma^p_t=0$	$s^a\phi^A=X^{Aa},\ s^bs^a\phi^A=\varepsilon^{ab}Y^A\ ,\ s^cs^bs^a\phi^A=0$
$\int dt'\ \frac{\delta X^{pa}_t}{\delta\Gamma^q_{t'}}X^{qb}_{t'}=\varepsilon^{ab}Y^p_t,\ Y^p_t=-\frac{1}{2}\varepsilon_{ab}\int dt'\frac{\delta X^{pa}_t}{\delta\Gamma^q_{t'}}X^{Bb}_{t'}$	$\frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb}=\varepsilon^{ab}Y^A,\ Y^A=-\frac{1}{2}\varepsilon_{ab}\frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb}$
$\int dt\ \frac{\delta X^{pa}_t}{\delta\Gamma^p_t}=\int dt'\ \frac{\delta Y^p_t}{\delta\Gamma^q_{t'}}X^{qa}_{t'}=0$	$\frac{\delta X^{Aa}}{\delta\phi^A}=\frac{\delta Y^A}{\delta\phi^B}X^{Bb}=0$
$\frac{\delta\left(\Delta\Gamma^p_{t'}\right)}{\delta\Gamma^q_{t''}}=M^p_{q\vert t',t''}=U^p_{q\vert t',t''}+V^p_{q\vert t',t''}+W^p_{q\vert t',t''}$	$\frac{\delta\left(\Delta\phi^A\right)}{\delta\phi^B}=M^A_B=P^A_B+Q^A_B+R^A_B$
$V^p_{q\vert t',t''}=(V_1)^p_{q\vert t',t''}+(V_2)^p_{q\vert t',t''}$	$Q^A_B=(Q_1)^A_B+(Q_2)^A_B$
$(V_1)^p_{q\vert t',t''}=\lambda_a\frac{\delta X^{pa}_{t'}}{\delta\Gamma^q_{t''}}(-1)^{\varepsilon_p+1}$	$(Q_1)^A_B=\lambda_a\frac{\delta X^{Aa}}{\delta\phi^B}(-1)^{\varepsilon_A+1}$
$(V_2)^p_{q\vert t',t''}=\lambda_aY^p_{t'}\frac{\delta\lambda^a}{\delta\Gamma^q_{t''}}(-1)^{\varepsilon_p+1}$	$(Q_2)^A_B=\lambda_aY^A\frac{\delta\lambda^a}{\delta\phi^B}(-1)^{\varepsilon_A+1}$
$U^p_{q\vert t',t''}=X^{pa}_{t'}\frac{\delta\lambda_a}{\delta\Gamma^q_{t''}}\ ,\ W^p_{q\vert t',t''}=-\frac{1}{2}\lambda^2\frac{\delta Y^p_{t'}}{\delta\Gamma^q_{t''}}$	$P^A_B=X^{Aa}\frac{\delta\lambda_a}{\delta\phi^B}\ ,\ R^A_B=-\frac{1}{2}\lambda^2\frac{\delta Y^A}{\delta\phi^B}$

Table 2: Correspondence of the matrix elements in arbitrary first-class constraint systems and Yang–Mills theories. Finite field-dependent BRST-antiBRST transformations with arbitrary parameters.

Proposition 5 *The matrices U, V, W possess the properties*

$$\mathrm{Str}\left(M^n\right)=\mathrm{Str}\left(U+V\right)^n+n\mathrm{Str}\left(U^{n-1}W\right)=\begin{cases}\mathrm{Str}\left(U+V\right)+\mathrm{Str}\left(W\right)\ , & n=1\ ,\\ \mathrm{Str}\left(U+V\right)^n\ , & n>1\ .\end{cases} \tag{4.51}$$

$$\mathrm{Str}\left(U+V\right)^n=\sum_{k=0}^{n}C^k_n\mathrm{Str}\left(U^{n-k}V^k\right)=\mathrm{Str}\left(U^n+C^1_nU^{n-1}V+C^2_nU^{n-2}V^2\right)\ ,\quad n=2,3\ , \tag{4.52}$$

$$\mathrm{Str}\left(U+V\right)^{2k}=\sum_{l=0}^{1}C^l_{2k}\mathrm{Str}\left(U^{2k-l}V^l\right)+C^1_{2k}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(U^lV\right)^2\right]+C^1_k\mathrm{Str}\left[\left(U^{k-1}V\right)^2\right],\quad k\geq 2\ , \tag{4.53}$$

$$\mathrm{Str}\left(U+V\right)^{2k+1}=\sum_{l=0}^{1}C^l_{2k+1}\mathrm{Str}\left(U^{2k+1-l}V^l\right)+C^1_{2k+1}\sum_{l=0}^{k-1}\mathrm{Str}\left[U^{2(k-l)-1}\left(U^lV\right)^2\right]\ ,\quad k\geq 2\ , \tag{4.54}$$

$$\mathrm{Str}\left(V_1\right)=0\ ,\quad \mathrm{Str}\left(W\right)-\frac{1}{2}\mathrm{Str}\left(V_1^2\right)=0\ , \tag{4.55}$$

$$\mathrm{Str}\left(U^n\right)=-\mathrm{tr}\left[(m^n)^a_b\right]\equiv-\mathrm{tr}\left(m^n\right)=-\left(m^n\right)^a_a\ ,\quad VU^n=\mathrm{tr}\left[m^{n-1}\left(e+m\right)Y\right]\ ,\quad n\geq 1\ ,\quad \text{where}\quad m^a_b\equiv s^a\lambda_b\ , \tag{4.56}$$

where $e=(e)^a_b\equiv\delta^a_b$, according to the notation of Subsection 4.1, and the matrix $Y=(Y^a_b)^p_{q|t',t''}$ is given by

$$\left(Y^a_b\right)^p_{q|t',t''}\equiv(-1)^{\varepsilon_p}\lambda^aY^p_{t',t''}\frac{\delta\lambda_b}{\delta\Gamma^q_{t''}}\Longrightarrow\left(Y^a_a\right)^p_{q|t',t''}=\left(V_2\right)^p_{q|t',t''}\ . \tag{4.57}$$

From (4.51)–(4.57), with allowance for $\mathrm{Str}\,(AB)=\mathrm{Str}\,(BA)$, it follows that $\Im$ acquires the form, cf. (4.16), (4.17),

$$\Im=-\mathrm{tr}\ln\left(e+m\right)\ ,\quad \text{where}\quad \ln\left[(e+m)\right]^a_b=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(m^n\right)^a_b\ . \tag{4.58}$$

The considerations of Subsection 4.1 following the relation (4.19) can now be repeated for the result (4.58), with account taken of the obvious replacement $\lambda_a\left(\phi\right)=s^a\Lambda\left(\phi\right)\rightarrow\lambda_a\left(\Gamma\right)=\int dt\ s_a\Lambda\left(\Gamma\right)$.

4.3 General Gauge Theories

The consideration of general gauge theories in Lagrangian formalism proves similar to the case of Yang–Mills theories and is based on the lemmas of Subsection 4.1, with minor modifications, necessary to take into account the facts that in

general gauge theories the global BRST-antiBRST transformations $\Gamma^{\mathsf{p}} \to \Gamma'^{\mathsf{p}} = \Gamma^{\mathsf{p}} + \delta\Gamma^{\mathsf{p}}$, $\delta\Gamma^{\mathsf{p}} = (\mathsf{s}^a\Gamma^{\mathsf{p}})\,\lambda_a$, $\lambda_a = \mathrm{const}$, do not respect the invariance of functional integration measure, $\Gamma'^{\mathsf{p}} \neq \Gamma^{\mathsf{p}}$, and do not possess the anticommutativity of the generators, $\mathsf{s}^a\mathsf{s}^b + \mathsf{s}^b\mathsf{s}^a \not\equiv 0$. Namely, in the general case the finite BRST-antiBRST transformations with arbitrary parameters $\lambda_a(\Gamma)$ have form

$$\Gamma^{\mathsf{p}} \to \Gamma'^{\mathsf{p}} = \Gamma^{\mathsf{p}} + \Delta\Gamma^{\mathsf{p}}\ , \quad \Delta\Gamma^{\mathsf{p}} = (\mathsf{s}^a\Gamma^{\mathsf{p}})\,\lambda_a + \frac{1}{4}\left(\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\lambda^2 = \mathcal{X}^{\mathsf{p}a}\lambda_a - \frac{1}{2}\mathcal{Y}^{\mathsf{p}}\lambda^2\ , \quad \lambda_a \not\equiv \mathsf{s}_a\Lambda\ .$$

Let us examine the corresponding even matrix $\mathcal{M} = \|\mathcal{M}^{\mathsf{p}}_{\mathsf{q}}\|$ and the related quantity $\Im$, namely,

$$\mathcal{M}^{\mathsf{p}}_{\mathsf{q}} = \frac{\delta\left(\Delta\Gamma^{p}\right)}{\delta\Gamma^{\mathsf{q}}}\ , \quad \Im = \mathrm{Str}\ln\left(\mathbb{I} + \mathcal{M}\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(\mathcal{M}^n\right)\ , \quad \mathbb{I}^{\mathsf{p}}_{\mathsf{q}} = \delta^{\mathsf{p}}_{\mathsf{q}}\ .$$

Explicitly, the matrix $\mathcal{M}^{\mathsf{p}}_{\mathsf{q}}$ is given by the sum of three even matrices:

$$\mathcal{M}^{\mathsf{p}}_{\mathsf{q}} = \frac{\delta\left(\Delta\Gamma^{p}\right)}{\delta\Gamma^{\mathsf{q}}} = \mathcal{U}^{\mathsf{p}}_{\mathsf{q}} + \mathcal{V}^{\mathsf{p}}_{\mathsf{q}} + \mathcal{W}^{\mathsf{p}}_{\mathsf{q}}\ , \ \text{ where } \ \mathcal{V}^{\mathsf{p}}_{\mathsf{q}} = \left(\mathcal{V}_1\right)^{\mathsf{p}}_{\mathsf{q}} + \left(\mathcal{V}_2\right)^{\mathsf{p}}_{\mathsf{q}}\ , \tag{4.59}$$

$$\mathcal{U}^{\mathsf{p}}_{\mathsf{q}} = \mathcal{X}^{\mathsf{p}a}\lambda_{a,q}\ , \quad \left(\mathcal{V}_1\right)^{\mathsf{p}}_{\mathsf{q}} = \lambda_a\mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\,(-1)^{\varepsilon_{\mathsf{p}}+1}\ , \quad \left(\mathcal{V}_2\right)^{\mathsf{p}}_{\mathsf{q}} = \lambda_a\mathcal{Y}^{\mathsf{p}}\lambda^a_{,\mathsf{q}}\,(-1)^{\varepsilon_{\mathsf{p}}+1}\ , \quad \mathcal{W}^{\mathsf{p}}_{\mathsf{q}} = -\frac{1}{2}\lambda^2\mathcal{Y}^{\mathsf{p}}_{,\mathsf{q}}\ , \tag{4.60}$$

The matrices $\mathcal{U}^{\mathsf{p}}_{\mathsf{q}}$, $\mathcal{V}^{\mathsf{p}}_{\mathsf{q}}$, $\mathcal{W}^{\mathsf{p}}_{\mathsf{q}}$ correspond to the matrices P^A_B, Q^A_B, R^A_B of Subsection 4.1. This correspondence is given explicitly by Table 3. In this connection, Lemmas 2, 4, 5 and the relations (4.4) of Subsection 4.1 remain formally

General gauge theories	Yang–Mills theories
$\Gamma^{\mathsf{p}}, \Delta\Gamma^{\mathsf{p}} = \mathcal{X}^{\mathsf{p}a}\lambda_a - (1/2)\,\mathcal{Y}^{\mathsf{p}}\lambda^2$	$\phi^A, \Delta\phi^A = X^{Aa}\lambda_a - (1/2)\,Y^A\lambda^2$
$\mathcal{Y}^{\mathsf{p}} = (1/2)\,\mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}\varepsilon_{ba}$	$Y^A = (1/2)\,X^{Aa}_{,B}X^{Bb}\varepsilon_{ba}$
$\frac{\delta(\Delta\Gamma^p)}{\delta\Gamma^{\mathsf{q}}} = \mathcal{M}^p_q$	$\frac{\delta(\Delta\phi^A)}{\delta\phi^B} = M^A_B$
$\mathcal{M}^{\mathsf{p}}_{\mathsf{q}} = \mathcal{U}^{\mathsf{p}}_{\mathsf{q}} + \mathcal{V}^{\mathsf{p}}_{\mathsf{q}} + \mathcal{W}^{\mathsf{p}}_{\mathsf{q}}$	$M^A_B = P^A_B + Q^A_B + R^A_B$
$\mathcal{V}^{\mathsf{p}}_{\mathsf{q}} = (\mathcal{V}_1)^{\mathsf{p}}_{\mathsf{q}} + (\mathcal{V}_2)^{\mathsf{p}}_{\mathsf{q}}$	$Q^A_B = (Q_1)^A_B + (Q_2)^A_B$
$(\mathcal{V}_1)^{\mathsf{p}}_{\mathsf{q}} = \lambda_a\mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\,(-1)^{\varepsilon_{\mathsf{p}}+1}$	$(Q_1)^A_B = \lambda_a X^{Aa}_{,B}\,(-1)^{\varepsilon_A+1}$
$(\mathcal{V}_2)^{\mathsf{p}}_{\mathsf{q}} = \lambda_a\mathcal{Y}^{\mathsf{p}}\lambda^a_{,\mathsf{q}}\,(-1)^{\varepsilon_{\mathsf{p}}+1}$	$(Q_2)^A_B = \lambda_a Y^A\lambda^a_{,B}\,(-1)^{\varepsilon_A+1}$
$\mathcal{U}^{\mathsf{p}}_{\mathsf{q}} = \mathcal{X}^{\mathsf{p}a}\lambda_{a,q}, \mathcal{W}^{\mathsf{p}}_{\mathsf{q}} = -\frac{1}{2}\lambda^2\mathcal{Y}^{\mathsf{p}}_{,\mathsf{q}}$	$P^A_B = X^{Aa}\lambda_{a,B}, R^A_B = -\frac{1}{2}\lambda^2 Y^A_{,B}$

Table 3: Correspondence of the matrix elements in Yang–Mills and general gauge theories. Finite field-dependent BRST-antiBRST transformations with arbitrary parameters.

the same in terms of $\mathcal{U}^{\mathsf{p}}_{\mathsf{q}}$, $\mathcal{V}^{\mathsf{p}}_{\mathsf{q}}$, $\mathcal{W}^{\mathsf{p}}_{\mathsf{q}}$, substituted instead of the respective matrices P^A_B, Q^A_B, R^A_B , since the relevant considerations do not use any properties of these objects, except their Grassmann parity and the character of their dependence on the parameters λ_a, which establishes the following

Proposition 6 *The matrices $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ possess the properties*

$$\mathrm{Str}\left(\mathcal{M}^{n}\right)=\mathrm{Str}\left(\mathcal{U}+\mathcal{V}\right)^{n}+n\mathrm{Str}\left(\mathcal{U}^{n-1}\mathcal{W}\right)\ ,\quad n\geq 1\ , \tag{4.61}$$

$$\mathrm{Str}\left(\mathcal{U}+\mathcal{V}\right)^{n}=\sum_{k=0}^{n}C_{n}^{k}\mathrm{Str}\left(\mathcal{U}^{n-k}\mathcal{V}^{k}\right)=\mathrm{Str}\left(\mathcal{U}^{n}+C_{n}^{1}\mathcal{U}^{n-1}\mathcal{V}+C_{n}^{2}\mathcal{U}^{n-2}\mathcal{V}^{2}\right)\ ,\quad n=2,3\ , \tag{4.62}$$

$$\mathrm{Str}\left(\mathcal{U}+\mathcal{V}\right)^{2k}=\sum_{l=0}^{1}C_{2k}^{l}\mathrm{Str}\left(\mathcal{U}^{2k-l}\mathcal{V}^{l}\right)+C_{2k}^{1}\sum_{l=0}^{k-2}\mathrm{Str}\left[\mathcal{U}^{2(k-l-1)}\left(\mathcal{U}^{l}\mathcal{V}\right)^{2}\right]+C_{k}^{1}\mathrm{Str}\left[\left(\mathcal{U}^{k-1}\mathcal{V}\right)^{2}\right],\quad k\geq 2\ , \tag{4.63}$$

$$\mathrm{Str}\left(\mathcal{U}+\mathcal{V}\right)^{2k+1}=\sum_{l=0}^{1}C_{2k+1}^{l}\mathrm{Str}\left(\mathcal{U}^{2k+1-l}\mathcal{V}^{l}\right)+C_{2k+1}^{1}\sum_{l=0}^{k-1}\mathrm{Str}\left[\mathcal{U}^{2(k-l)-1}\left(\mathcal{U}^{l}\mathcal{V}\right)^{2}\right]\ ,\quad k\geq 2\ , \tag{4.64}$$

$$\mathrm{Str}\left(\mathcal{U}^{n}\right)=-\mathrm{tr}\left[(\mathsf{m}^{n})^{a}_{\ b}\right]\equiv-\mathrm{tr}\left(\mathsf{m}^{n}\right)=-\left(\mathsf{m}^{n}\right)^{a}_{\ a}\ ,\quad \text{where}\quad \mathsf{m}^{a}_{b}\equiv\mathsf{s}^{a}\lambda_{b}\ , \tag{4.65}$$

$$\mathcal{V}\mathcal{U}^{n}=\mathrm{tr}\left[\mathsf{m}^{n-1}\left(e+\mathsf{m}\right)\mathcal{Y}\right]\ ,\quad n\geq 1\ , \tag{4.66}$$

where $e=(e)^{a}_{b}\equiv\delta^{a}_{b}$, according to the notation of Subsection 4.1, and the matrix $\mathcal{Y}=(\mathcal{Y}^{b}_{a})^{\mathsf{p}}_{\mathsf{q}}$ is given by

$$(\mathcal{Y}^{a}_{b})^{\mathsf{p}}_{\mathsf{q}}\equiv(-1)^{\varepsilon_{\mathsf{p}}}\lambda^{a}\mathcal{Y}^{\mathsf{p}}\frac{\delta\lambda_{b}}{\delta\Gamma^{\mathsf{q}}}\Longrightarrow(\mathcal{Y}^{a}_{a})^{\mathsf{p}}_{\mathsf{q}}=(\mathcal{V}_{2})^{\mathsf{p}}_{\mathsf{q}}\ . \tag{4.67}$$

On the other hand, Lemmas 1, 3 use the explicit structure of functions entering the matrices $(Q_1)^{A}_{B}$, R^{A}_{B} and they consequently undergo, in terms of $(\mathcal{V}_1)^{\mathsf{p}}_{\mathsf{q}}$, $\mathcal{W}^{\mathsf{p}}_{\mathsf{q}}$, the following modifications, established in respective Appendices B.10, B.11:

Lemma 9 *There hold the properties*

$$\mathrm{Str}\left(\mathcal{U}^{n-1}\mathcal{W}\right)=-\frac{1}{4}\frac{\delta\lambda_{a}}{\delta\Gamma^{\mathsf{p}}}\left(\mathsf{m}^{n-2}\right)^{a}_{b}\left(\mathrm{s}^{b}\mathrm{s}^{2}\Gamma^{\mathsf{p}}\right)\lambda^{2}\ ,\quad n>1\ . \tag{4.68}$$

Lemma 10 *The matrices $\mathcal{V}_1$ and $\mathcal{W}$ are related by the equality*

$$\mathrm{Str}\left(\mathcal{V}_{1}\right)+\mathrm{Str}\left(\mathcal{W}\right)-\frac{1}{2}\mathrm{Str}\left(\mathcal{V}_{1}^{2}\right)=-\left(\Delta^{a}S\right)\lambda_{a}-\frac{1}{4}\left(\mathsf{s}_{a}\Delta^{a}S\right)\lambda^{2}\ . \tag{4.69}$$

Note: the properties in (4.68) generalize the equalities $\mathrm{Str}\left(P^{n-1}R\right)=0$, $n>2$, implied by (4.5), due to the failure of the generators s^{a} to be nilpotent in the entire space Γ^{p}, which means that the matrix $\mathcal{W}$ does not drop out of $\mathrm{Str}\left(\mathcal{M}^{n}\right)$, $n>1$; the relation (4.69) extends the properties (4.11) to the case of non-anticommuting generators s^{a} and a BRST-antiBRST non-invariant integration measure $d\Gamma$, and has been established in our paper [4]; for the sake of completeness of the present subsection, the corresponding proof is given in Appendix B.11.

In view of the properties (4.61)–(4.67) and the correspondence provided by Table 3, the calculation of the quantity $\Im$ here repeats the considerations of Appendix B.6, with the modifications provided by (4.68), (4.69), in comparison with (4.5), (4.11), which implies the appearance in $\Im$ of an extra contribution:

$$\mathrm{Str}\left(\mathcal{V}_{1}\right)+\mathrm{Str}\left(\mathcal{W}\right)-\frac{1}{2}\mathrm{Str}\left(\mathcal{V}_{1}^{2}\right)-\sum_{n=2}^{\infty}\frac{(-1)^{n}}{n}n\mathrm{Str}\left(\mathcal{U}^{n-1}\mathcal{W}\right)\ . \tag{4.70}$$

Thus, the resulting expression for $\Im$ is given by, cf. (4.16),

$$\Im=-\left(\Delta^{a}S\right)\lambda_{a}-\frac{1}{4}\left(\mathsf{s}_{a}\Delta^{a}S\right)\lambda^{2}-\mathrm{tr}\ln\left(e+\mathsf{m}\right)+\Re\ , \tag{4.71}$$

where

$$\Re=\frac{1}{4}\lambda_{a,\mathsf{p}}[(e+\mathsf{m})^{-1}]^{a}_{b}\left(\mathrm{s}^{b}\mathrm{s}^{2}\Gamma^{\mathsf{p}}\right)\lambda^{2}\ , \tag{4.72}$$

or, explicitly,

$$[\ln(e+\mathsf{m})]^a_b = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\mathsf{m}^n)^a_b\ , \quad [(e+\mathsf{m})^{-1}]^a_b = \sum_{n=0}^{\infty}(-1)^n(\mathsf{m}^n)^a_b\ . \tag{4.73}$$

In contrast to our paper [4], the result expressed by (4.71), (4.72) makes no assumption that the functional parameters λ_a do not depend on some of their variables from the set $\Gamma^{\mathsf{p}} = \left(\phi^A, \pi^{Aa}, \lambda^A, \phi^*_{Aa}, \bar{\phi}_A\right)$, namely, that λ_a are restricted to $(\phi^A, \pi^{Aa}, \lambda^A)$, making it thereby possible to utilize the anticommutativity of the BRST-antiBRST generators s^a in this subspace. This restriction has now been removed due to the fact that the considerations of Appendix B.6 do not require, as has been noticed in Subsection 4.1, the BRST-antiBRST generators to be anticommuting, except for the treatment of the terms $\mathrm{Str}(R)$, $\mathrm{Str}\left(Q_1^2\right)$, $\mathrm{Str}\left(P^{n-1}R\right)$, which now correspond to part of the contribution (4.70). At the same time, by virtue of (4.68), (4.69), this contribution has now been calculated for non-anticommuting generators s^a, thereby extending the considerations of Appendix B.6 to general gauge theories. As a consequence, the result expressed by (4.71), (4.72) is now presented in terms of arbitrary anticommuting parameters $\lambda_a(\Gamma)$. In this connection, let us examine the change of the integrand corresponding to the result (4.71), (4.72):

$$d\Gamma \exp\left[(i/\hbar)\mathcal{S}_F(\Gamma)\right]\big|_{\Gamma\to\Gamma'} = d\Gamma\exp\left\{(i/\hbar)\left[\mathcal{S}_F(\Gamma+\Delta\Gamma) - i\hbar\Im(\Gamma)\right]\right\}\ ,$$

where, taking into account the relation (B.101) of Appendix B.11, we have

$$\begin{aligned}
&\mathcal{S}_F(\Gamma+\Delta\Gamma) = \mathcal{S}_F(\Gamma) + \Delta\mathcal{S}_F(\Gamma)\ ,\\
&\Delta\mathcal{S}_F = (\mathsf{s}^a\mathcal{S}_F)\lambda_a + \frac{1}{4}(\mathsf{s}_a\mathsf{s}^a\mathcal{S}_F)\lambda^2 = -i\hbar\Delta^a S\lambda_a - \frac{i\hbar}{4}\mathsf{s}_a(\Delta^a S)\lambda^2\ ,\\
&\Im = -(\Delta^a S)\lambda_a - \frac{1}{4}(\mathsf{s}_a\Delta^a S)\lambda^2 - \mathrm{tr}\ln(e+\mathsf{m}) + \frac{1}{4}\lambda_{a,\mathsf{p}}[(e+\mathsf{m})^{-1}]^a_b\left(\mathrm{s}^b\mathrm{s}^2\Gamma^{\mathsf{p}}\right)\lambda^2\ ,
\end{aligned} \tag{4.74}$$

whence

$$d\Gamma\exp\left(\frac{i}{\hbar}\mathcal{S}_F\right)\bigg|_{\Gamma\to\Gamma'} = d\Gamma\exp\left\{\frac{i}{\hbar}\left[\mathcal{S}_F + i\hbar\,\mathrm{tr}\ln(e+\mathsf{m}) - \frac{i\hbar}{4}\lambda_{a,\mathsf{p}}[(e+\mathsf{m})^{-1}]^a_b\left(\mathrm{s}^b\mathrm{s}^2\Gamma^{\mathsf{p}}\right)\lambda^2\right]\right\} \equiv d\Gamma\exp\left(\frac{i}{\hbar}\mathcal{S}'\right)\ , \tag{4.75}$$

which implies that, due to the presence in $\mathcal{S}'$ of a non-vanishing contribution with λ^2, the corresponding modified quantum action $\mathcal{S}'$ generally does not describe a change of gauge-fixing:

$$\mathcal{S}' \not\equiv S + \phi^*_{Aa}\pi^{Aa} + \bar{\phi}_A\lambda^A - \frac{1}{2}U^2F'\ , \quad U^a = \mathrm{s}^a\big|_{\phi,\pi,\lambda}\ . \tag{4.76}$$

If we now require that $\mathcal{S}' = S + \phi^*_{Aa}\pi^{Aa} + \bar{\phi}_A\lambda^A - (1/2)U^2F'$ be indeed the case, then there arise the conditions

$$\lambda_{a,\mathsf{p}}[(e+\mathsf{m})^{-1}]^a_b\left(\mathrm{s}^b\mathrm{s}^2\Gamma^{\mathsf{p}}\right) = 0\ , \tag{4.77}$$

$$i\hbar\,\mathrm{tr}\ln(e+\mathsf{m}) = -(1/2)U^2\Delta F\ , \quad \Delta F = (F'-F)\ . \tag{4.78}$$

If we furthermore assume that $\Delta F = \Delta F(\phi,\pi,\lambda)$, which in the case $\Delta F = \Delta F(\phi)$ represents a change of the gauge in the Sp(2)-covariant scheme [12, 13], then the r.h.s. and l.h.s. of (4.78) are independent of the antifields ϕ^*_{Aa}, $\bar{\phi}_A$,

$$\frac{\delta\lambda_a}{\delta\phi^*_{Aa}} = \frac{\delta\lambda_a}{\delta\bar{\phi}_A} = 0\ ,$$

implying that the condition (4.77) is thereby fulfilled:

$$\begin{aligned}
&\lambda_{a,\mathsf{p}}[(e+\mathsf{m})^{-1}]^a_b\left(\mathrm{s}^b\mathrm{s}^2\Gamma^{\mathsf{p}}\right) = \lambda_{a,\underline{\mathsf{p}}}[(e+\mathsf{m})^{-1}]^a_b\left(\mathrm{s}^b\mathrm{s}^2\Gamma^{\underline{\mathsf{p}}}\right) = 0\ ,\\
&\Gamma^{\mathsf{p}} = \left(\Gamma^{\underline{\mathsf{p}}}, \Gamma^{\overline{\mathsf{p}}}\right)\ , \quad \Gamma^{\underline{\mathsf{p}}} = \left(\phi^A, \pi^{Aa}, \lambda^A\right)\ , \quad \Gamma^{\overline{\mathsf{p}}} = \left(\phi^*_{Aa}, \bar{\phi}_A\right)\ ,
\end{aligned} \tag{4.79}$$

because the generators s^a are nilpotent in the subspace $\Gamma_{\underline{\mathrm{p}}} = \left(\phi^A, \pi^{Aa}, \lambda^A\right)$, namely, $\mathrm{s}^a \mathrm{s}^2 \Gamma_{\underline{\mathrm{p}}} = U^a U^2 \Gamma_{\underline{\mathrm{p}}} \equiv 0$. The remaining condition (4.78) therefore acquires the form

$$\Im = \frac{1}{2i\hbar} U^2 \Delta F \ , \quad \text{where} \quad \Im = -\mathrm{tr} \ln \left(e + \tilde{\mathsf{m}}\right) \ , \quad \left(\tilde{\mathsf{m}}\right)^a_b \equiv U^a \lambda_b \ . \tag{4.80}$$

Due to the anticommutativity of U^a, we can now make use of Lemma 6 of Subsection 4.1 with a subsequent criterion which can now be represented as follows: the quantity $\mathrm{tr} \ln \left(e + \mathsf{m}\right)$ is BRST-antiBRST-exact (U^a-exact) to all orders of its expansion in powers of λ_a if and only if there exists such an even-valued functional Λ that

$$U^a \lambda_a = -U^2 \Lambda \Longrightarrow \lambda_a = U_a \Lambda + \psi_a \ , \quad U^a \psi_a = 0 \ . \tag{4.81}$$

Thus, supposing that (4.81) is indeed the case and taking account of (4.28), (4.29), in terms of U^a replacing s^a, we find that the condition (4.80) is satisfied and reads equivalently

$$\ln \left(1 - \frac{1}{2} U^2 \Lambda\right)^{-2} - \frac{1}{4} U^2 \left(\psi^2\right) \left[\left(1 - \frac{1}{2} U^2 \Lambda\right)^{-2}\right] = \frac{1}{2i\hbar} U^2 \Delta F \ , \quad \psi^2 \equiv \psi_a \psi^a \ , \tag{4.82}$$

which is a compensation equation that expresses Λ in terms of a gauge variation ΔF and a certain solution ψ_a to the equation $U^a \psi_a = 0$. The relation (4.82) can be accompanied by comments similar to those which follow (4.29). In the particular case $U^2 \left(\psi^2\right) = 0$, the relation (4.82) reduces to the usual compensation equation.

5 Relating Gauges in Standard Model and Gribov Ambiguity

Let us consider an application of finite field-dependent BRST-antiBRST transformations to a fundamental physical model describing almost the entire variety of the known elementary particles. Namely, we examine the Lagrangian description of the Standard Model [50, 51, 52, 54, 55, 56, 69, 70, 71], which is an example of a Yang–Mills theory interacting with spinor and scalar fields.

The classical non-renormalized action of the Standard Model in Minkowski space-time is given by the sum of several contributions:

$$S_{\mathrm{SM}} = \int d^4 x \ \mathcal{L}_{\mathrm{SM}} \ , \quad \mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\text{gauge fields}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{quarks}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Higgs}} \ , \tag{5.1}$$

where the Lagrangian density for the even-valued gauge fields $\mathcal{A}^m_\mu(x) = (A_\mu, A^{\hat{a}}_\mu, A^{\underline{\alpha}}_\mu)(x)$ has the form[10]

$$\begin{aligned} &\mathcal{L}_{\text{gauge fields}} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} F^{\hat{a}}_{\mu\nu} F^{\mu\nu\hat{a}} - \frac{1}{4} F^{\underline{\alpha}}_{\mu\nu} F^{\mu\nu\underline{\alpha}} \ , \\ &f_{\mu\nu} = \partial_{[\mu} A_{\nu]} \ , \quad F^{\hat{a}}_{\mu\nu} = \partial_{[\mu} A^{\hat{a}}_{\nu]} + g \varepsilon^{\hat{a}}_{\hat{b}\hat{c}} A^{\hat{b}}_\mu A^{\hat{c}}_\nu \ , \quad F^{\underline{\alpha}}_{\mu\nu} = \partial_{[\mu} A^{\underline{\alpha}}_{\nu]} + g_{\mathrm{s}} f^{\underline{\alpha}}_{\underline{\beta\gamma}} A^{\underline{\beta}}_\mu A^{\underline{\gamma}}_\nu \ , \\ &A_\mu \in u(1) \ , \quad A^{\hat{a}}_\mu \tau_{\hat{a}} \in su(2) \ , \quad A^{\underline{\alpha}}_\mu \lambda_{\underline{\alpha}} \in su(3) \ , \end{aligned} \tag{5.2}$$

with $\tau_{\hat{a}}$, $\hat{a} = 1, 2, 3$, and $(1/2)\, \lambda_{\underline{\alpha}}$, $\underline{\alpha} = 1, \ldots, 8$, being the $su(2)$ Pauli matrices and Hermitian traceless Gell-Mann matrices, satisfying the $su(2)$ and $su(3)$ commutation relations

$$[\tau_{\hat{a}}, \tau_{\hat{a}}] = 2i \varepsilon_{\hat{a}\hat{b}\hat{c}} \tau_{\hat{c}} \ , \quad \left[\frac{1}{2} \lambda_{\underline{\alpha}}, \frac{1}{2} \lambda_{\underline{\beta}}\right] = i f_{\underline{\alpha\beta\gamma}} \frac{1}{2} \lambda_{\underline{\gamma}} \ . \tag{5.3}$$

The Lagrangian for the odd-valued leptons l^k_L, l^k_R, being Dirac spinors, reads as follows:

$$\mathcal{L}_{\text{leptons}} = \sum_{k=1}^{3} \left[\bar{l}^k_L i \gamma^\mu \left(\partial_\mu - i \frac{g}{2} A^{\hat{a}}_\mu \tau_{\hat{a}} + i \frac{g'}{2} A_\mu\right) l^k_L + \bar{l}^k_R i \gamma^\mu \left(\partial_\mu + i g' A_\mu\right) l^k_R\right] \ , \tag{5.4}$$

[10] The field A_μ is not to be confused with the electromagnetic potential.

where l_L^k are left-handed $SU(2)$-doublets, l_R^k are right-handed $SU(2)$-singlets, g, $g_{\rm s}$, g' are the coupling constants, and γ^μ are the 4×4 Dirac matrices subject to the normalization $\{\gamma^\mu,\, \gamma^\nu\} = 2\eta^{\mu\nu}$, with the Minkowski metric tensor $\eta^{\mu\nu}$. The quantities[11] $\varepsilon^{\hat{a}}_{\hat{b}\hat{c}} = \varepsilon_{\hat{a}\hat{b}\hat{c}} = \varepsilon^{\hat{a}\hat{b}\hat{c}}$ and $f^{\underline{\alpha}}_{\underline{\beta\gamma}} = f_{\underline{\alpha\beta\gamma}} = f^{\underline{\alpha\beta\gamma}}$ in (5.2) and (5.3) are completely antisymmetric.

The QCD (quark) sector of the strong interactions described by the 3 quark generations (u, d), (c, s), (t, b), organized into Dirac spinors $(u_k, d_k)^T$, has the form

$$\mathcal{L}_{\rm quarks} = \sum_{k=1}^{3}\left\{\begin{bmatrix}\bar{u}_k\\ \bar{d}'_k\end{bmatrix}_L i\gamma^\mu\left[\partial_\mu - i\frac{g_{\rm s}}{2}A^{\underline{\alpha}}_\mu\lambda_{\underline{\alpha}} - i\frac{g}{2}A^{\hat{a}}_\mu\tau_{\hat{a}} - i\frac{g'}{6}A_\mu\right]\begin{bmatrix}u_k\\ d'_k\end{bmatrix}_L\right.$$
$$\left. + \,\bar{u}^k_R i\gamma^\mu\left[\partial_\mu - i\frac{g_{\rm s}}{2}A^{\underline{\alpha}}_\mu\lambda_{\underline{\alpha}} - i\frac{2g'}{3}A_\mu\right]u^k_R + \bar{d}'^k_R i\gamma^\mu\left[\partial_\mu - i\frac{g_{\rm s}}{2}A^{\underline{\alpha}}_\mu\lambda_{\underline{\alpha}} + i\frac{g'}{3}A_\mu\right]d'^k_R\right\}\ , \tag{5.5}$$

$$d'^k = U^{kk'}_{\rm CKM}d^{k'}\ ,\quad u^k = (u, c, t)\ ,\quad d^k = (d, s, b)\ , \tag{5.6}$$

where the respective left- and right-handed $SU(3)$-triplets $(u_k, d_k)^T_L$ and $(u_k, d_k)^T_R$, are $SU(2)$-doublets and $SU(2)$-singlets, respectively, and $U_{\rm CKM}$ is the Cabibbo–Kobayashi–Maskawa matrix [73].

The masses of particles in the Standard Model are generated by the Yukawa interaction term

$$\mathcal{L}_{\rm Yukawa} = -\frac{1}{\sqrt{2}}\sum_{k=1}^{3}\left\{f^u_k\begin{bmatrix}\bar{u}^k\\ \bar{d}^k\end{bmatrix}_L\varphi u^k_R + f^d_k\begin{bmatrix}\bar{u}^k\\ \bar{d}^k\end{bmatrix}_L\varphi d^k_R + f^l_k\bar{l}^k_L\varphi l^k_R + \text{h.c.}\right\}\ , \tag{5.7}$$

where f^u_k, f^d_k, f^l_k are the Yukawa couplings, and the Brout–Englert–Higgs Lagrangian is given by

$$\mathcal{L}_{\rm Higgs} = \frac{1}{2}\left|\left(i\partial_\mu + (g/2)\,A^{\hat{a}}_\mu\tau_{\hat{a}} + (g'/2)\,A_\mu\right)\right|\varphi^2 - \frac{\mu^2}{2}|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4\ , \tag{5.8}$$

where φ is a Bosonic field, being an $SU(2)$-doublet, μ^2 is a negative constant, and λ is the Higgs self-interaction coupling constant.

We consider the minimal Standard Model, which means that the neutrinos entering the left-handed $SU(2)$ doublets l^k_L are assumed to be massless.

The action $S_{\rm SM}$ in (5.1) is invariant with respect to the following gauge transformations acting in the configuration space $\mathcal{M}_{\rm SM}$:

$$\mathcal{M}_{\rm SM} = A^i = \{\mathcal{A}^m_\mu;\Sigma^I\}(x) = \left\{\mathcal{A}^m_\mu; l^{k\hat{A}}_L, \bar{l}^{k\hat{B}}_L, l^k_R, \bar{l}^k_R, ((u_k,d_k)^p_L)^T, ((\bar{u}_k,\bar{d}_k)^q_L)^T, \left((u_k,d_k)^{\underline{A}}_R\right)^T, \left((\bar{u}_k,\bar{d}_k)^{\underline{B}}_R\right)^T, \varphi^{\hat{C}}\right\}(x)\ , \tag{5.9}$$

where $[\hat{A};\hat{A}';\hat{C}] = [(\hat{a},1);(\hat{a}',1');(\hat{c},1)]$, $[\underline{A};\underline{B}] = [(\underline{\alpha},1);(\underline{\beta},1)]$, and Dirac-conjugated spinors, such as $\bar{l}^k_L$, are assumed to be independent:

$$\delta\left(\mathcal{A}^{\mu m},\Sigma^I\right)(x) = \int d^4y\ \left(R^{\mu m}_n, R^I_n\right)(x,y)\varsigma^n(y)\ ,\quad \alpha = (n,y)\ ,\quad \varsigma^n = \left(\varsigma, \varsigma^{\hat{b}}, \varsigma^{\underline{\beta}}\right)\ .$$

Here, the generators $\left(R^{\mu m}_n, R^I_n\right)(x,y) = \left(R^{\mu m}_n(\mathcal{A}), R^I_n(\Sigma)\right)\delta(x-y)$ form the Lie algebra of the gauge transformations and read as follows: for the gauge fields $\mathcal{A}^{\mu m}$,

$$R^{\mu m}_n(\mathcal{A}) = \begin{cases}\partial^\mu\ , & m = 1,\ n = 1,\\ \partial^\mu\delta^{\hat{a}\hat{b}} + g\varepsilon^{\hat{a}\hat{c}\hat{b}}A^{\mu\hat{c}}\ , & m = \hat{a},\ n = \hat{b},\\ \partial^\mu\delta^{\underline{\alpha\beta}} + g_s f^{\underline{\alpha\gamma\beta}}A^{\mu\underline{\gamma}}\ , & m = \underline{\alpha},\ n = \underline{\beta},\end{cases} \tag{5.10}$$

[11] The explicit form of the Gell-Mann matrices $\lambda_{\underline{\alpha}}$, as well as the $su(3)$ structure constants $f_{\underline{\alpha\beta\gamma}}$, may be found, e.g., in [72].

and for the matter fields Σ^I,

$$R^I_n(\Sigma) = \begin{cases} \left(0,\ -g\varepsilon^{\hat{a}\hat{c}\hat{b}}(l^k_L)^{\hat{c}},\ \frac{1}{2}g' l^k_L\right)(x), & I=(\hat{a},1),\ n=(\underline{\beta},\hat{b},1), \\ \left(0,\ -g\varepsilon^{\hat{a}'\hat{c}\hat{b}}(\bar{l}^k_L)^{\hat{c}},\ \frac{1}{2}g' \bar{l}^k_L\right)(x), & I=(\hat{a}',1^{\mathrm{I}}), \\ (0,\ 0,\ g' l^k_R)(x), & I=1^{\mathrm{II}}, \\ (0,\ 0,\ g' \bar{l}^k_R)(x), & I=1^{\mathrm{III}}, \\ \left(-g_s f^{\underline{\alpha\gamma\beta}}\left((u_k,d_k)^T_L\right)^{\underline{\gamma}},\ -g\varepsilon^{\hat{d}\hat{e}\hat{b}}\left((u_k,d_k)^T_L\right)^{\hat{e}},\ -\frac{1}{6}g'(u_k,d_k)^T_L\right)(x), & I=(\underline{\alpha},\hat{d},1^{\mathrm{IV}}), \\ \left(-g_s f^{\underline{\alpha'\gamma\beta}}\left((\bar{u}_k,\bar{d}_k)^T_L\right)^{\underline{\gamma}},\ -g\varepsilon^{\hat{d}'\hat{e}\hat{b}}\left((\bar{u}_k,\bar{d}_k)^T_L\right)^{\hat{e}},\ -\frac{1}{6}g'(\bar{u}_k,\bar{d}_k)^T_L\right)(x), & I=(\underline{\alpha}',\hat{d}',1^{\mathrm{V}}), \\ \left(-g_s f^{\underline{\delta\sigma\beta}}(u_{k\,R})^{\underline{\sigma}},\ 0,\ -\frac{2}{3}g' u_{k\,R}\right)(x), & I=(\underline{\delta},1^{\mathrm{VI}}), \\ \left(-g_s f^{\underline{\delta'\sigma\beta}}(d_{k\,R})^{\underline{\sigma}},\ 0,\ \frac{1}{3}g' d_{k\,R}\right)(x), & I=(\underline{\delta}',1^{\mathrm{VII}}), \\ \left(0,\ g\varepsilon^{\hat{c}\hat{e}\hat{b}}\varphi^{\hat{e}},\ \frac{1}{2}g'\varphi\right)(x), & I=(\hat{c},1^{\mathrm{VIII}}), \end{cases} \tag{5.11}$$

with $k = 1,2,3$ and the structure constants F^{lmn} in the sector of the gauge fields $\mathcal{A}^m_\mu$ given by

$$F^{\gamma}_{\alpha\beta} = F^{lmn}\delta(x-z)\delta(y-z)\ , \quad F^{lmn} = \left(0,\ g\varepsilon^{\hat{a}\hat{b}\hat{c}},\ g_s f^{\underline{\alpha\beta\gamma}}\right)\ , \tag{5.12}$$

for the $U(1)$, $SU(2)$, $SU(3)$ gauge subgroups, respectively. The form of the structure constants for the given model is obviously consistent with (2.9), taking account of the convention

$$D^{mn}_\mu = \delta^{mn}\partial_\mu + F^{mln}\mathcal{A}^l_\mu\ . \tag{5.13}$$

In (5.11), we do not expose the explicit structure of the Dirac spinor indices, implying that it enters the index I, except for the gauge and Higgs fields. Besides, the scalar indices 1 and $1^{\mathrm{I}}, \ldots, 1^{\mathrm{VIII}}$ correspond to the $U(1)$ group of the weak hypercharge.

Under the assumption that the vacuum expectation values of all the fields are zero, we present the Higgs field φ as follows:

$$\varphi = \begin{bmatrix} 0 \\ \eta + \chi \end{bmatrix} + i\zeta^a\tau_a \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \zeta_2 + i\zeta_1 \\ \eta + \chi - i\zeta_3 \end{bmatrix}\ , \tag{5.14}$$

where ζ_a are the Goldstone Bosons, η is the vacuum expectation value of the Higgs field, and χ are fluctuations of the Higgs field.

Let us choose a gauge Boson $F_{\boldsymbol{\xi}}$ corresponding to an $R_{\boldsymbol{\xi}}$-like family of gauges, parameterized by a set of numbers $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and related to the Landau and Feynman (covariant) gauges for $\boldsymbol{\xi} = (0,0,0)$ and $\boldsymbol{\xi} = (1,1,1)$, respectively:

$$F_{\boldsymbol{\xi}}\left(\mathcal{A}, C\right) = -\frac{1}{2}\int d^4x\ \left(A_\mu A^\mu + A^{\hat{a}}_\mu A^{\hat{a}\mu} + A^{\underline{\alpha}}_\mu A^{\underline{\alpha}\mu}\right) + \frac{1}{4}\varepsilon_{ab}\int d^4x\ \left(\xi_1 C^a C^b + \xi_2 C^{\hat{a}a}C^{\hat{a}b} + \xi_3 C^{\underline{\alpha}a}C^{\underline{\alpha}b}\right)\ . \tag{5.15}$$

Using (2.2), (2.6)–(2.9), we can now present the corresponding quantum action $S_F(A)$ in the path integral (2.1). In doing so, we extend the results of [1], considering the part that deals with the Yang–Mills theory, in the sense that the relevant formulae[12] are now written down in the specific cases of the $U(1)$, $SU(2)$, $SU(3)$ groups and feature contributions related to the presence of all the three cases, complete with the corresponding classical fields $(A_\mu, A^{\hat{a}}_\mu, A^{\underline{\alpha}}_\mu)$, as well as the ghost-antighost $\left(C^a, C^{\hat{a}a}, C^{\underline{\alpha}a}\right)$ and Nakanishi–Lautrup $\left(B, B^{\hat{a}}, B^{\underline{\alpha}}\right)$ fields. The quantum action $S_F(A,B,C)$ corresponding to the gauge-fixing functional (5.15) reads

$$S_{F_{\boldsymbol{\xi}}}(A,B,C) = S_{\mathrm{SM}}\left(A\right) + (1/2)\, s^a s_a \left[F_{\boldsymbol{\xi}}\left(\mathcal{A}, C\right)\right] = S_{\mathrm{SM}}\left(A\right) + S_{\mathrm{gf}}\left(\mathcal{A}, B; \boldsymbol{\xi}\right) + S_{\mathrm{gh}}\left(\mathcal{A}, C; \boldsymbol{\xi}\right) + S_{\mathrm{add}}\left(C; \boldsymbol{\xi}\right)\ , \tag{5.16}$$

[12] Specifically, we use Eqs. (4.11)–(4.16) of [1], where f^{lmn} are identified with F^{lmn} in (5.12).

where the gauge-fixing term S_{gf}, the ghost term S_{gh}, and the interaction term S_{add}, quartic in the ghost-antighost fields, are given by

$$
\begin{aligned}
S_{\mathrm{gf}} &= \int d^4x\ \left\{\left[(\partial^\mu A_\mu) + \frac{\xi_1}{2}B\right]B + \left[\left(\partial^\mu A^{\hat{a}}_\mu\right) + \frac{\xi_2}{2}B^{\hat{a}}\right]B^{\hat{a}} + \left[\left(\partial^\mu A^{\underline{\alpha}}_\mu\right) + \frac{\xi_3}{2}B^{\underline{\alpha}}\right]B^{\underline{\alpha}}\right\}\ , && (5.17)\\
S_{\mathrm{gh}} &= \frac{1}{2}\varepsilon_{ab}\int d^4x\ \left[(\partial^\mu C^a)\,\partial_\mu C^b + \left(\partial^\mu C^{\hat{a}a}\right)D^{\hat{a}\hat{b}}_\mu C^{\hat{b}b} + \left(\partial^\mu C^{\underline{\alpha}a}\right)D^{\underline{\alpha\beta}}_\mu C^{\underline{\beta}b}\right]\ , && (5.18)\\
S_{\mathrm{add}} &= -\frac{1}{48}\varepsilon_{ab}\varepsilon_{cd}\int d^4x\ \left(\xi_2 g^2\varepsilon^{\hat{a}\hat{b}\hat{e}}\varepsilon^{\hat{e}\hat{c}\hat{d}}C^{\hat{d}a}C^{\hat{c}c}C^{\hat{b}b}C^{\hat{a}d} + \xi_3 g_s^2 f^{\underline{\alpha\beta\sigma}}f^{\underline{\sigma\gamma\delta}}C^{\underline{\delta}a}C^{\underline{\gamma}c}C^{\underline{\beta}b}C^{\underline{\alpha}d}\right)\ . && (5.19)
\end{aligned}
$$

In (5.15), the gauge-fixing functional $F_{\mathbf{0}}\left(\mathcal{A},C\right)$, with $\boldsymbol{\xi} = (0,0,0)$,

$$
F_{\mathbf{0}}\left(\mathcal{A},C\right) = -\frac{1}{2}\int d^4x\ \left(A_\mu A^\mu + A^{\hat{a}}_\mu A^{\hat{a}\mu} + A^{\underline{\alpha}}_\mu A^{\underline{\alpha}\mu}\right)\ , \qquad (5.20)
$$

induces the contribution $S_{\mathrm{gf}}\left(\mathcal{A},B\right)$ to the quantum action that arises in the case of the Landau gauge, $\chi^m\left(\mathcal{A}\right) = \partial^\mu\mathcal{A}^m_\mu$, whereas the functional $F_{\mathbf{1}}\left(\mathcal{A},C\right)$, with $\boldsymbol{\xi} = (1,1,1)$,

$$
F_{\mathbf{1}}\left(\mathcal{A},C\right) = -\frac{1}{2}\int d^4x\ \left(A_\mu A^\mu + A^{\hat{a}}_\mu A^{\hat{a}\mu} + A^{\underline{\alpha}}_\mu A^{\underline{\alpha}\mu}\right) + \frac{1}{4}\varepsilon_{ab}\int d^4x\ \left(C^a C^b + C^{\hat{a}a}C^{\hat{a}b} + C^{\underline{\alpha}a}C^{\underline{\alpha}b}\right)\ , \qquad (5.21)
$$

corresponds to the Feynman gauge, $\chi^m\left(\mathcal{A},B\right) = \partial^\mu\mathcal{A}^m_\mu + (1/2)\,B^m$.

In order to find the parameters $\lambda_a = s_a\Lambda$ of a finite field-dependent BRST-antiBRST transformation that connects an $R_{\boldsymbol{\xi}}$ gauge with an $R_{\boldsymbol{\xi}+\Delta\boldsymbol{\xi}}$ gauge, according to (2.24), we need the quantities $\Delta F_{\boldsymbol{\xi}}$, $s^a\left(\Delta F_{\boldsymbol{\xi}}\right)$, $s^a s_a\left(\Delta F_{\boldsymbol{\xi}}\right)$, which are evaluated using

$$
\begin{aligned}
&\Delta F_{\boldsymbol{\xi}} = F_{\boldsymbol{\xi}+\Delta\boldsymbol{\xi}} - F_{\boldsymbol{\xi}} = \frac{1}{4}\varepsilon_{ab}\int d^4x\ \left(\Delta\xi_1 C^a C^b + \Delta\xi_2 C^{\hat{a}a}C^{\hat{a}b} + \Delta\xi_3 C^{\underline{\alpha}a}C^{\underline{\alpha}b}\right)\ ,\\
&s^a\left(\Delta F_{\boldsymbol{\xi}}\right) = \frac{1}{2}\int d^4x\ \left(\Delta\xi_1 B C^a + \Delta\xi_2 B^{\hat{a}}C^{\hat{a}a} + \Delta\xi_3 B^{\underline{\alpha}}C^{\underline{\alpha}a}\right)\ , \qquad (5.22)\\
&\frac{1}{2}s^a s_a\left(\Delta F_{\boldsymbol{\xi}}\right) = \left(S_{\mathrm{gf}} + S_{\mathrm{gh}} + S_{\mathrm{add}}\right)\big|_{\boldsymbol{\xi}+\Delta\boldsymbol{\xi}} - \left(S_{\mathrm{gf}} + S_{\mathrm{gh}} + S_{\mathrm{add}}\right)\big|_{\boldsymbol{\xi}}\ ,
\end{aligned}
$$

with allowance made for (2.7), (5.12) and $B^m = \left(B, B^{\hat{a}}, B^{\underline{\alpha}}\right)$, $C^{na} = \left(C^a, C^{\hat{a}a}, C^{\underline{\alpha}a}\right)$,

$$
\begin{aligned}
&s^a A^i(x) = X^{ia}_1(x) = \left(R^{\mu m}_n(\mathcal{A}), R^I_n(\Sigma)\right)C^{na}(x)\ ,\\
&-\frac{1}{2}s^2 A^i(x) = Y^i_1(x) = \left(R^{\mu m}_n(\mathcal{A}), R^I_n(\Sigma)\right)B^n(x) - \frac{1}{2}\varepsilon_{ab}\left(F^{mrn}C^{ra}R^{\mu n}_s C^{sb}, \frac{\partial R^I_m(\Sigma)}{\partial\Sigma^J}R^J_n(\Sigma)C^{na}C^{mb}\right)(x)\ ,\\
&s^a B^m(x) = X^{ma}_2(x) = -\frac{1}{2}F^{mln}\left(B^n C^{la} + \frac{1}{6}F^{nrs}C^{sb}C^{ra}C^{lc}\varepsilon_{cb}\right)(x)\ ,\\
&-\frac{1}{2}s^2 B^m(x) = Y^m_2(x) = 0\,,\\
&s^a C^{mb}(x) = X^{mab}_3(x) = -\left(\varepsilon^{ab}B^m + \frac{1}{2}F^{mln}C^{nb}C^{la}\right)(x)\ ,\\
&-\frac{1}{2}s^2 C^{ma}(x) = Y^{ma}_3(x) = -2X^{ma}_2(x)\ , \qquad (5.23)
\end{aligned}
$$

which determines the finite BRST-antiBRST transformations $\phi^A \to \phi'^A = \phi^A\exp(\overleftarrow{s}^a\lambda_a)$ in the Standard Model.

As a result, the functional parameters $\lambda_a = s_a\Lambda$ that connect an $R_{\boldsymbol{\xi}}$-like gauge to an $R_{\boldsymbol{\xi}+\Delta\boldsymbol{\xi}}$-like gauge are given

by an extension of the result [1], featuring the contributions related to all the three groups $U(1)$, $SU(2)$, $SU(3)$:

$$\lambda_a = \frac{1}{4i\hbar}\varepsilon_{ab}\int d^4x\ \left(\Delta\xi_1 BC^b + \Delta\xi_2 B^{\hat{a}}C^{\hat{a}b} + \Delta\xi_3 B^{\underline{\alpha}}C^{\underline{\alpha}b}\right)$$
$$\times\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left\{\frac{1}{4i\hbar}\int d^4y\ \left[\Delta\xi_1 B^2 + \Delta\xi_2\left(B^{\hat{a}}B^{\hat{a}} - \frac{g^2}{24}\varepsilon^{\hat{a}\hat{b}\hat{e}}\varepsilon^{\hat{e}\hat{c}\hat{d}}C^{\hat{d}c}C^{\hat{c}e}C^{\hat{b}d}C^{\hat{a}g}\varepsilon_{cd}\varepsilon_{eg}\right)\right.\right.$$
$$\left.\left. +\ \Delta\xi_3\left(B^{\underline{\alpha}}B^{\underline{\alpha}} - \frac{g_s^2}{24}f^{\underline{\alpha\beta\sigma}}f^{\underline{\sigma\gamma\delta}}C^{\underline{\delta}c}C^{\underline{\gamma}e}C^{\underline{\beta}d}C^{\underline{\alpha}g}\varepsilon_{cd}\varepsilon_{eg}\right)\right]\right\}^n\ , \tag{5.24}$$

where the corresponding potential Λ is given by

$$\Lambda = \frac{1}{8i\hbar}\varepsilon_{ab}\int d^4x\ \left(\Delta\xi_1 C^aC^b + \Delta\xi_2 C^{\hat{a}a}C^{\hat{a}b} + \Delta\xi_3 C^{\underline{\alpha}a}C^{\underline{\alpha}b}\right)\times$$
$$\times\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left\{\frac{1}{4i\hbar}\int d^4y\ \left[\Delta\xi_1 B^2 + \Delta\xi_2\left(B^{\hat{a}}B^{\hat{a}} - \frac{g^2}{24}\varepsilon^{\hat{a}\hat{b}\hat{e}}\varepsilon^{\hat{e}\hat{c}\hat{d}}C^{\hat{d}c}C^{\hat{c}e}C^{\hat{b}d}C^{\hat{a}g}\varepsilon_{cd}\varepsilon_{eg}\right)\right.\right.$$
$$\left.\left. +\ \Delta\xi_3\left(B^{\underline{\alpha}}B^{\underline{\alpha}} - \frac{g_s^2}{24}f^{\underline{\alpha\beta\sigma}}f^{\underline{\sigma\gamma\delta}}C^{\underline{\delta}c}C^{\underline{\gamma}e}C^{\underline{\beta}d}C^{\underline{\alpha}g}\varepsilon_{cd}\varepsilon_{eg}\right)\right]\right\}^n\ . \tag{5.25}$$

This solves the problem of reaching any gauge in the family of $R_{\boldsymbol{\xi}}$-like gauges, starting from a certain gauge encoded in the path integral by a functional $F_{\boldsymbol{\xi}}$, within the BRST-antiBRST quantization of the Standard Model, by means of finite BRST-antiBRST transformations with field-dependent parameters λ_a.

According to (2.1), the generating functionals of Green's functions for the Standard Model in $R_{\boldsymbol{\xi}}$-like gauges read as follows:

$$Z_{\mathrm{SM},\boldsymbol{\xi}}(J,\eta) = \int d\widetilde{\phi}\ \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{SM}}(\widetilde{A}) - \frac{1}{2}F_{\boldsymbol{\xi}}\overleftarrow{s}^2 + J_A\widetilde{\phi}^A\right]\right\} = \exp\left[\frac{i}{\hbar}W_{\mathrm{SM},\boldsymbol{\xi}}(J,\eta)\right]\ , \tag{5.26}$$

$$\exp\left[\frac{i}{\hbar}\Gamma_{\mathrm{SM},\xi}(\phi,\eta)\right] = \int d\widetilde{\phi}\ \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{SM}}(\widetilde{A}) - \frac{1}{2}F_{\boldsymbol{\xi}(\widetilde{A},\widetilde{C})}\overleftarrow{s}^2 + \frac{\delta\Gamma_{\mathrm{SM},\xi}(\phi)}{\delta\phi^A}\left(\phi^A - \widetilde{\phi}^A\right)\right]\right\}\ , \tag{5.27}$$

where the effective action[13] $\Gamma_{\mathrm{SM},\boldsymbol{\xi}}(\phi,\eta)$ is the Legendre transform of W_{SM} with respect to J_A, namely,

$$\Gamma_{\mathrm{SM},\boldsymbol{\xi}}(\phi,\eta) = W_{\mathrm{SM},\boldsymbol{\xi}}(J,\eta) - J_A\phi^A\ , \quad \text{where}\quad \phi^A = \frac{\delta W_{\mathrm{SM}}}{\delta J_A}\ ,\quad \frac{\delta\Gamma_{\mathrm{SM}}}{\delta\phi^A} = -J_A\ . \tag{5.28}$$

The modified Ward identity (2.25) for $Z_{\mathrm{SM},\boldsymbol{\xi}}(J,\eta)$ depends on field-dependent parameters, $\lambda_a(\Delta\boldsymbol{\xi}) = \Lambda(\Delta\boldsymbol{\xi})\overleftarrow{s}_a$ in (5.24), and has the form

$$\left\langle\left\{1 + \frac{i}{\hbar}J_A\left[X^{Aa}\lambda_a(\Delta\boldsymbol{\xi}) - \frac{1}{2}Y^A\lambda^2(\Delta\boldsymbol{\xi})\right] - \frac{1}{4}\left(\frac{i}{\hbar}\right)^2\varepsilon_{ab}J_AX^{Aa}J_BX^{Bb}\lambda^2(\Delta\boldsymbol{\xi})\right\}\left(1 - \frac{1}{2}\Lambda(\Delta\boldsymbol{\xi})\overleftarrow{s}^2\right)^{-2}\right\rangle_{F_{\boldsymbol{\xi}},J} = 1\ . \tag{5.29}$$

The non-Abelian nature of the gauge group, because of the differential gauges [75] implied by the gauge Boson (5.15), leads to the Gribov ambiguity [44], described initially in the Coulomb gauge, and controlled in the Gribov–Zwanziger theory [45, 46] by using the horizon functionals $h_{\mathbf{0}}$ and $h_{\mathbf{1}}$ in the Landau and Feynman gauges, respectively. For contemporary considerations, justified by lattice calculations of Gribov copies, see, e.g., [76, 77, 78]. For applications of the Gribov–Zwanziger theory in the Coulomb, Landau and maximal Abelian gauges, as well as in covariant $R_{\boldsymbol{\xi}}$-gauges in the pure Yang–Mills theory, see [79, 80, 81, 82, 83, 84, 85, 86, 87, 88]. Notice that there exist other approaches intended to eliminate (or bypass) the Gribov ambiguity problem: first, the procedure of imposing an algebraic (instead of differential) gauge on auxiliary scalar fields in a theory which is non-perturbatively equivalent

[13] The minimal Standard Model on a nontrivial gravitational background with $g_{\mu\nu}(x) = \eta_{\mu\nu} + \ldots$ has been examined, e.g., in [74], where the effective action, depending on $g_{\mu\nu}$ and η, was determined on the mass shell.

to the Yang–Mills theory [89, 90, 91], second, the procedure of averaging over the Gribov copies with a non-uniform weight in the path integral and the replica trick [92, 93], third, the incorporation of the Gribov factor (restricting the functional measure in the path integral to the first Gribov region) into the Faddeev–Popov matrix, thereby modifying the gauge algebra of gauge transformations [34].

As we turn to the Gribov ambiguity problem and Gribov–Zwanziger theory, it should be noted, first of all, that the Landau gauge implies, due to the preservation of the gauge condition when extracting the unique representative from the gauge orbit of field configurations in terms of the equation

$$\partial_\mu\left(\mathcal{A}^{\mu m}(x)+\delta\mathcal{A}^{\mu m}(x)\right)=\partial_\mu\mathcal{A}^{\mu m}(x)\Longrightarrow\int d^4y\ \partial_\mu R^{\mu m}_n(x,y)\varsigma^n(y)=\partial_\mu\left(\partial^\mu\varsigma,D^{\hat{a}\hat{b}}_\mu\varsigma^{\hat{b}},D^{\underline{\alpha\beta}}_\mu\varsigma^{\underline{\beta}}\right)(x)=(0,0,0)\ , \quad (5.30)$$

that, in addition to a vanishing solution $\varsigma^n_0(x)=(0,0,0)$, there also exist many smooth solutions $\varsigma^n_{(k_1,k_2)}(x)=\left(0,\varsigma^{\hat{b}}_{k_1},\varsigma^{\underline{\beta}}_{k_2}\right)(x)$ for configurations of the non-Abelian gauge fields $\mathcal{A}^{\mu\hat{a}},\mathcal{A}^{\mu\underline{\alpha}}$ vanishing at the spatial infinity in Minkowski space-time. Second, the Gribov–Zwanziger theory implies the sum of the horizon functionals corresponding to the $SU(2)$ and $SU(3)$ gauge groups[14]:

$$h_{\mathbf{0}}(\mathcal{A})=h^{SU(2)}_{\mathbf{0}}+h^{SU(3)}_{\mathbf{0}}\ , \quad (5.31)$$

$$h^{SU(2)}_{\mathbf{0}}=\gamma_1^2g^2\int d^4x\ d^4y\ \varepsilon^{\hat{a}\hat{b}\hat{c}}A^{\hat{b}}_\mu(x)\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{d}}(x;y)\,\varepsilon^{\hat{d}\hat{e}\hat{c}}A^{\mu\hat{e}}(y)+4\cdot3\gamma_1^2g^2\ , \quad (5.32)$$

$$h^{SU(3)}_{\mathbf{0}}=\gamma_2^2g_s^2\int d^4x\ d^4y\ f^{\underline{\alpha\beta\gamma}}A^{\underline{\beta}}_\mu(x)\left(K^{-1}_{SU(3)}\right)^{\underline{\alpha\delta}}(x;y)\,f^{\underline{\delta\sigma\gamma}}A^{\mu\underline{\sigma}}(y)+4\cdot8\gamma_2^2g_s^2\ , \quad (5.33)$$

where $h_{\mathbf{0}}$ does not depend on the matter fields Σ^I, and $K^{-1}_{SU(N)}$, $N=2,3$, is the inverse,

$$\int d^4z\ \left(K^{-1}_{SU(N)}\right)^{ml}(x;z)\left(K_{SU(N)}\right)^{ln}(z;y)=\int d^4z\ \left(K^{-1}_{SU(N)}\right)^{nl}(x;z)\left(K_{SU(N)}\right)^{lm}(z;y)=\delta^{mn}\delta\left(x-y\right)\ , \quad (5.34)$$

of the (Hermitian) Faddeev–Popov operator, $K^{mn}_{SU(N)}=\partial^\mu D^{mn}_\mu$, induced by the gauge-fixing functional $F_{\mathbf{0}}$. Here, the thermodynamic ("Gribov mass") parameters γ_1 and γ_2 of [45, 46] are introduced in a self-consistent way using the gap equations for the functional $S_{F_{\mathbf{0}},h}$, being the Gribov–Zwanziger action in the BRST-antiBRST approach to the Standard Model,

$$\frac{\partial}{\partial\gamma_i}\left\{\frac{\hbar}{i}\ln\left[\int d\phi\ \exp\left(\frac{i}{\hbar}S_{F_{\mathbf{0}},h}\right)\right]\right\}=\frac{\partial\mathcal{E}_{\rm vac}}{\partial\gamma_i}=0\ ,\quad\text{where}\quad i=1,2\ . \quad (5.35)$$

Here, we have used the definition of the vacuum energy $\mathcal{E}_{\rm vac}$ and introduced a modified quantum action for the Gribov–Zwanziger model as an extension of the Yang–Mills quantum action $S_{F_{\mathbf{0}}}$ in (5.16), using the Landau gauge:

$$S_{F_{\mathbf{0}},h}\left(\phi\right)=S_{\rm SM}(A)-\frac{1}{2}\left(F_{\mathbf{0}}\overleftarrow{s}^2\right)(\phi)+h_{\mathbf{0}}(\mathcal{A})\,, \quad (5.36)$$

The action $S_{F_{\mathbf{0}},h}\left(\phi\right)$ is non-invariant under the finite BRST-antiBRST transformations:

$$S_{F_{\mathbf{0}},h}(\phi\exp[\overleftarrow{s}^a\lambda_a])=S_{F_{\mathbf{0}},h}(\phi)+\Delta h(\phi)=S_{F_{\mathbf{0}},h}(\phi)+\left(h_{\mathbf{0}}\overleftarrow{s}^a\lambda_a\right)(\phi)+\frac{1}{4}\left(h_{\mathbf{0}}\overleftarrow{s}^2\lambda^2\right)(\phi)\neq S_{F_{\mathbf{0}},h}(\phi)\ . \quad (5.37)$$

The covariant gauge implies two options: one of them preserves the gauge independence of the conventional S-matrix, according to the BRST-antiBRST extension [1] of the Gribov–Zwanziger theory, and the other one determines the horizon functional in terms of transverse-like non-Abelian gauge fields [64] (see, as well [94]). Let us examine the

[14] Further on, the consideration of the Gribov–Zwanziger theory is based on the assumption that we deal with the $\mathbb{R}^4$ Euclidean space-time.

first option, which implies a finite BRST-antiBRST-transformed functional $h_{\mathbf{0}}(\mathcal{A})$:

$$\begin{aligned} h_{\boldsymbol{\xi}}(\mathcal{A},B,C) &= h_{\boldsymbol{\xi}}^{SU(2)}(\mathcal{A},B,C) + h_{\boldsymbol{\xi}}^{SU(3)}(\mathcal{A},B,C)\ , \\ h_{\boldsymbol{\xi}}(\mathcal{A},B,C) &= h_{\mathbf{0}} + \frac{1}{2i\hbar}\left(s^a h_{\mathbf{0}}\right)\left(s_a \Delta F_{\boldsymbol{\xi}}\right)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}s^b s_b \Delta F_{\boldsymbol{\xi}}\right)^n \\ &\qquad - \frac{1}{16\hbar^2}\left(s^2 h_{\mathbf{0}}\right)\left(s\Delta F_{\boldsymbol{\xi}}\right)^2\left[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}s^b s_b \Delta F_{\boldsymbol{\xi}}\right)^n\right]^2\ . \end{aligned} \tag{5.38}$$

Here, $s^a h_{\mathbf{0}} = s^a\left(h_{\mathbf{0}}^{SU(2)} + h_{\mathbf{0}}^{SU(3)}\right)$ are given by the following expressions, with account taken of the definition (5.12) for the structure constants F^{lmn}:

$$\begin{aligned} s^a h_{\mathbf{0}}^{SU(2)} &= \gamma_1^2 g^2 \varepsilon^{\hat{a}\hat{b}\hat{c}}\varepsilon^{\hat{c}\hat{d}\hat{e}}\int d^4x\ d^4y\ \left[2D_\mu^{\hat{b}\hat{p}}C^{\hat{p}a}(x)\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{d}}(x;y)\right. \\ &\quad \left. - g\varepsilon^{\hat{q}\hat{r}\hat{s}}\int d^4x'\ d^4y'\ A_\mu^{\hat{b}}(x)\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{q}}(x;x')\,K_{SU(2)}^{\hat{r}\hat{u}}(x';y')\,C^{\hat{u}a}(y')\left(K^{-1}_{SU(2)}\right)^{\hat{s}\hat{d}}(y';y)\right]A^{\hat{e}\mu}(y)\ , \end{aligned} \tag{5.39}$$

$$\begin{aligned} s^a h_{\mathbf{0}}^{SU(3)} &= \gamma_2^2 g_s^2 f^{\underline{\alpha\beta\gamma}} f^{\underline{\gamma\delta\epsilon}}\int d^4x\ d^4y\ \left[2D_\mu^{\underline{\beta\rho}}C^{\underline{\rho}a}(x)\left(K^{-1}_{SU(3)}\right)^{\underline{\alpha\delta}}(x;y)\right. \\ &\quad \left. - g_s f^{\underline{\sigma\varsigma\tau}}\int d^4x'\ d^4y'\ A_\mu^{\underline{\beta}}(x)\left(K^{-1}_{SU(3)}\right)^{\underline{\alpha\sigma}}(x;x')\,K_{SU(3)}^{\underline{\varsigma\upsilon}}(x';y')\,C^{\underline{\tau}a}(y')\left(K^{-1}_{SU(3)}\right)^{\underline{\upsilon\delta}}(y';y)\right]A^{\underline{\epsilon}\mu}(y)\ , \end{aligned} \tag{5.40}$$

whereas $s^2 h_{\mathbf{0}} = s^2\left(h_{\mathbf{0}}^{SU(2)} + h_{\mathbf{0}}^{SU(3)}\right)$ is given by

$$\begin{aligned} s^2 h_{\mathbf{0}} &= s^2 h_{\mathbf{0}}^{SU(3)} + \gamma_1^2 g^2 \varepsilon^{\hat{a}\hat{b}\hat{c}}\varepsilon^{\hat{c}\hat{d}\hat{e}}\int d^4x\ d^4y\ \left\{4\left(-D_\mu^{\hat{c}\hat{p}}B^{\hat{p}} + \frac{g}{2}\varepsilon^{\hat{c}\hat{p}\hat{q}}C^{\hat{q}a}D_\mu^{\hat{p}\hat{r}}C^{\hat{r}b}\varepsilon_{ab}\right)(x)\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{d}}(x;y)\,A^{\hat{e}\mu}(y)\right. \\ &\quad + 2\varepsilon_{ab}D_\mu^{\hat{c}\hat{p}}C^{\hat{p}a}(x)\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{d}}(x;y)\,D^{\hat{e}\hat{q}\mu}C^{\hat{q}b}(y) \\ &\quad - 4\varepsilon_{ab}g\varepsilon^{\hat{p}\hat{q}\hat{r}}\int d^4x'\ d^4y'\ D_\mu^{\hat{c}\hat{s}}C^{\hat{s}a}(x)\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{p}}(x;x')\,K_{SU(2)}^{\hat{q}\hat{u}}(x';y')\,C^{\hat{u}b}(y')\left(K^{-1}_{SU(2)}\right)^{\hat{r}\hat{d}}(y';y)\,A^{\hat{e}\mu}(y) \\ &\quad + g\varepsilon^{\hat{p}\hat{q}\hat{r}}\int d^4x'\ d^4y'\ A_\mu^{\hat{c}}(x)\left[-g\varepsilon_{ab}\varepsilon^{\hat{s}\hat{u}\hat{v}}\int d^4x''\ d^4y''\ \left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{s}}(x;x'')\,K_{SU(2)}^{\hat{u}\hat{w}}(x'';y'')\,C^{\hat{w}a}(y'')\right. \\ &\quad \times\left(K^{-1}_{SU(2)}\right)^{\hat{v}\hat{p}}(y'';x')\,K_{SU(2)}^{\hat{q}\hat{g}}(x';y')\,C^{\hat{g}b}(y')\left(K^{-1}_{SU(2)}\right)^{\hat{r}\hat{d}}(y';y) - g\varepsilon_{ab}\varepsilon^{\hat{q}\hat{u}\hat{v}}\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{p}}(x;x')\,K_{SU(2)}^{\hat{v}\hat{w}}(x';y') \\ &\quad \times C^{\hat{w}a}(y')C^{\hat{u}b}(x')\left(K^{-1}_{SU(2)}\right)^{\hat{r}\hat{d}}(y';y) + 2\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{p}}(x;x')\,K_{SU(2)}^{\hat{q}\hat{g}}(x';y')\,B^{\hat{g}}(y')\left(K^{-1}_{SU(2)}\right)^{\hat{r}\hat{d}}(y';y) \\ &\quad + g\varepsilon_{ab}\varepsilon^{\hat{s}\hat{u}\hat{v}}\left(K^{-1}_{SU(2)}\right)^{\hat{a}\hat{p}}(x;x')\,K_{SU(2)}^{\hat{q}\hat{g}}(x';y')\,C^{\hat{g}a}(y') \\ &\quad \left.\left.\times\int d^4x''\ d^4y''\ \left(K^{-1}_{SU(2)}\right)^{\hat{r}\hat{s}}(y';x'')\,K_{SU(2)}^{\hat{u}\hat{w}}(x'';y'')\,C^{\hat{w}b}(y'')\left(K^{-1}_{SU(2)}\right)^{\hat{v}\hat{d}}(y'';y)\right]A^{\hat{e}\mu}(y)\right\}\ . \end{aligned} \tag{5.41}$$

Here, $s^2 h_{\mathbf{0}}^{SU(3)}$ has the same form as $s^2 h_{\mathbf{0}}^{SU(2)}$, in which one makes a replacement of the expressions $g\varepsilon^{\hat{a}\hat{b}\hat{c}}$, $(K,K^{-1})_{SU(2)}$ and the $SU(2)$-indices by the expressions $g_s f^{\underline{\alpha\beta\gamma}}$, $(K,K^{-1})_{SU(3)}$ and the $SU(3)$-indices, respectively. The quantities $s_a\Delta F_{\boldsymbol{\xi}}$, $s^a s_a \Delta F_{\boldsymbol{\xi}}$ and $\lambda^a_{\boldsymbol{\xi}}(\phi)$ in (5.38) are given by (5.22) and (5.24) for $\boldsymbol{\Delta\xi} = \boldsymbol{\xi}$, which relates the Landau gauge to an arbitrary R_ξ-like gauge,

$$Z_{\mathrm{SM},h_{\boldsymbol{\xi}}} = \int d\phi\ \exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{SM}} - \frac{1}{2}F_{\boldsymbol{\xi}}\overleftarrow{s}^2 + h_{\mathbf{0}}\exp\left(\overleftarrow{s}^a\lambda_{\boldsymbol{\xi}|a}\right)\right]\right\}\ , \tag{5.42}$$

in a manner respecting the gauge-independence of the corresponding S-matrix. In turn, the modified Ward identities (2.25) for the generating functional $Z_{\mathrm{SM},h_{\mathbf{0}}}(J,\eta)$ are obtained in the same way as for the generating functional

$Z_{\mathrm{SM},\boldsymbol{\xi}}(J,\eta)$ (5.26) without the horizon functional (5.43),

$$\left\langle\left\{1+\frac{i}{\hbar}\left[J_A\phi^A+h_{\mathbf{0}}\right]\left[\overleftarrow{s}^a\lambda_a(\boldsymbol{\xi})+\frac{1}{4}\overleftarrow{s}^2\lambda^2(\boldsymbol{\xi})\right]-\frac{1}{4}\left(\frac{i}{\hbar}\right)^2\left[J_A\phi^A+h_{\mathbf{0}}\right]\overleftarrow{s}^a\left[J_B\phi^B+h_{\mathbf{0}}\right]\overleftarrow{s}_a\lambda^2(\boldsymbol{\xi})\right\}\right.$$
$$\left.\left(1-\frac{1}{2}\Lambda\left(\boldsymbol{\xi}\right)\overleftarrow{s}^2\right)^{-2}\right\rangle_{h_{\mathbf{0}},F_{\mathbf{0}},J}=1\ . \tag{5.43}$$

which are reduced, at constant parameters λ_a, to an Sp(2)-doublet of the usual Ward identities (at the first order in λ_a), as well as to a derivative identity (at the second order in λ_a),

$$\left\langle\left(J_A+h_{\mathbf{0},A}\right)X^{Aa}\right\rangle_{h_{\mathbf{0}},F_{\mathbf{0}},J}=0\ ,\quad \left\langle\left(J_A+h_{\mathbf{0},A}\right)\left[2Y^A+(i/\hbar)\,\varepsilon_{ab}X^{Aa}\left(J_B+h_{\mathbf{0},B}\right)X^{Bb}\right]\right\rangle_{h_{\mathbf{0}},F_{\mathbf{0}},J}=0\ , \tag{5.44}$$

for the non-renormalized Standard Model in the Gribov–Zwanziger approach. Here, the symbol "$\langle\mathcal{O}\rangle_{h_{\mathbf{0}},F_{\mathbf{0}},J}$" for a quantity $\mathcal{O}=\mathcal{O}(\phi)$ denotes a source-dependent average expectation value with respect to $Z_{\mathrm{SM},h_{\mathbf{0}}}(J,\eta)$ corresponding to a gauge-fixing $F_{\mathbf{0}}$:

$$\langle\mathcal{O}\rangle_{h_{\mathbf{0}},F_{\mathbf{0}},J}=Z^{-1}_{\mathrm{SM},h_{\mathbf{0}}}(J,\eta)\int d\phi\ \mathcal{O}\left(\phi\right)\exp\left\{\frac{i}{\hbar}\left[S_{\mathrm{SM}}-\frac{1}{2}F_{\mathbf{0}}\overleftarrow{s}^2+h_{\mathbf{0}}+J_A\phi^A\right]\right\}\ ,\quad \text{with}\quad \langle 1\rangle_{h_{\mathbf{0}},F_{\mathbf{0}},J}=1\ . \tag{5.45}$$

The modified and standard Ward identities for Green's functions are readily obtained from (5.43) and (5.44), respectively, using differentiation over the sources. These identities are fulfilled in the tree approximation and provide a basis for the study of pa renormalization procedure using an appropriate gauge-invariant regularization. We intend to study this problem in separate research.

6 Discussion

We have extended the results and ideas of our previous study [1, 2, 3, 4] and have also applied them to the Lagrangian description of the Standard Model. The main results of the present study are given by Sections 3, 4, devoted to the calculation of functional Jacobians, which requires only the definition of such a Jacobian and does not have recourse to functional integration in itself. We have proposed and applied an explicit recipe of exact calculation of the Jacobian for a change of variables in the vacuum functional corresponding to finite field-dependent BRST-antiBRST transformations with a linear dependence on functionally-dependent parameters in Yang–Mills theories and first-class constraint dynamical systems, given, respectively, in Sections 3.1 and 3.2, by the relations (3.9), (3.10) and (3.30)–(3.32). This implies that thus linearized finite BRST-antiBRST transformations can be interpreted neither as global symmetry transformations of the integrand, nor as field-dependent transformations inducing an exact change of the gauge-fixing functional, despite the hope of the authors of [27]; see Eqs. (3.1)–(3.7) therein. At the same time, we have evaluated the Jacobian for a change of variables in the vacuum functional corresponding to finite field-dependent BRST-antiBRST transformations with *arbitrary* functional parameters $\lambda_a(\phi)\not\equiv s_a\Lambda(\phi)$, in Yang–Mills theories, first-class constraint dynamical systems, and general gauge theories, (4.16), (4.17), (4.58), (4.71), (4.72), which is the main result of the present work. It is demonstrated that the Jacobians are reduced to the previously known Jacobians in the case of functionally-dependent odd-valued parameters $\lambda_a=s_a\Lambda$, whereas in general gauge theories the Jacobian (4.71) has been obtained for the first time. We have demonstrated that in the general case $\lambda_a\not\equiv s_a\Lambda$ (more exactly, $s^a\lambda_a\neq s^a s_a\Lambda$) the Jacobian fails to be BRST-antiBRST-invariant, which implies the inconsistency of the compensation equation with such odd-valued parameters, and thereby entails the appearance, under such a change of variables, of terms which cannot be absorbed into a change of the gauge Boson, used in [15, 16] to provide the consistency of the compensation equations by using a suitable choice of the parameters in a functionally-dependent form. We have found that the set of functionally-dependent parameters $\lambda_a=s_a\Lambda$ generated by an s_a-gradient of an Sp(2)-scalar Λ can be

extended by an s_a-divergence of a symmetric Sp(2)-tensor $\Psi_{\{ab\}}$, namely, $\lambda_a = s_a\Lambda + s^b\Psi_{\{ab\}}$, which, as shown by (4.28), (4.40) in Yang–Mills theories and by (4.82) in general gauge theories, produces a non-trivial contribution to the Jacobians, thereby modifying the compensation equations, (4.29), (4.82), and affecting the change of the respective gauge Boson. In Yang–Mills theories, we have found the solutions (4.30)–(4.32) of the modified compensation equation (4.29), in particular, a non-trivial solution (4.31) which induces a zero change of gauge-fixing, $\Delta F = 0$, resulting in a Jacobian equal to unity, $\exp(\Im) = 1$. We have also presented (4.47) the BRST-antiBRST-non-exact contribution $\Re$ to the Jacobian induced by finite BRST-antiBRST transformations with arbitrary functional parameters. This contribution is to be regarded as an extra part of the transformed quantum action in the integrand (4.48). The same holds true for general gauge theories, in view of (4.75), (4.76).

Having applied our results [1] to the evaluation of Jacobians in the case of Yang–Mills theories, we have explicitly constructed the functionally-dependent parameters λ_a in (5.24) induced by a finite change $\Delta F_{\boldsymbol{\xi}}$ of the gauge Boson (5.22) in the quantum action of the Standard Model (5.16), which generates a change of the gauge in the path integral within a class of linear 3-parameter $R_{\boldsymbol{\xi}}$-like gauges, realized in terms of the even-valued gauge functionals $F_{\boldsymbol{\xi}}$ in (5.15), with the values $(\xi_1, \xi_2, \xi_3) = \mathbf{0}, \mathbf{1}$ corresponding to the Landau and Feynman (covariant) gauges, respectively. We have obtained a *modified* Ward identity (5.29) for a generating functional of Green's functions $Z_{\mathrm{SM},\boldsymbol{\xi}}(J,\eta)$ depending on field-dependent parameters, $\lambda_a = \Lambda \overleftarrow{s}_a$, which reduces to the usual Ward identity for a constant doublet λ_a.

In order to eliminate residual gauge invariance, i.e., Gribov copies, and to determine a consistent path integral for the Standard Model in the entire set of field configurations, we have explicitly constructed the Gribov–Zwanziger theory in the BRST-antiBRST Lagrangian description of the Standard Model. The construction extends the quantum action in the Landau gauge by a BRST-antiBRST non-invariant horizon functional $h_{\mathbf{0}}$ in (5.31)–(5.33). We have found the horizon functional $h_{\boldsymbol{\xi}}$ given by (5.38)–(5.41) in arbitrary $R_{\boldsymbol{\xi}}$-like gauges by means of field-dependent BRST-antiBRST transformations with the parameters λ_a given by (5.24) and providing the gauge-independence of the conventional S-matrix related to the Gribov–Zwanziger path integral $Z_{\mathrm{SM},h_{\boldsymbol{\xi}}}(\eta)$ in (5.42). We have obtained the modified (5.43) and usual (5.44) Ward identities for the generating functional of Green's functions $Z_{\mathrm{SM},h_{\mathbf{0}}}(J,\eta)$ providing a basis for renormalization. These are the main results of Section 5.

As has been noticed in Introduction and Section 5, there remains another option to determine the horizon functional in covariant $R_{\boldsymbol{\xi}}$-like gauges, which lies in transverse-like non-Abelian gauge fields [64], recently examined also in [94]. Namely, the Faddeev–Popov operators $K_{SU(N)}(\boldsymbol{\xi})$, $N = 2,3$, retain the same formal structure at any values of the gauge parameters $\boldsymbol{\xi}$. With this in mind, let us consider some extensions $\bar{K}^{\hat{a}\hat{b}}_{i|SU(2)}$, $\bar{K}^{\underline{\alpha\beta}}_{i|SU(3)}$, for $i = 1, 2$, of the Faddeev–Popov operators in $R_{\boldsymbol{\xi}}$-like gauges (5.15),

$$\bar{K}^{\hat{a}\hat{b}}_{i|SU(2)}(A,B;\xi_2) = K^{\hat{a}\hat{b}}_{SU(2)} + \frac{g\xi_2}{2}\varepsilon^{\hat{a}\hat{c}\hat{b}}\left[\delta_{i2}\partial_\mu\left(\frac{\partial^\mu B^{\hat{c}}}{\partial^2}\right) + \frac{\delta_{i1}}{2}B^{\hat{c}}\right] \ , \text{ where } \left(\bar{K}^{\hat{a}\hat{b}}_i(\xi_2)\right)^\dagger = \bar{K}^{\hat{a}\hat{b}}_i(\xi_2)^{15} \ ,$$
$$\bar{K}^{\underline{\alpha\beta}}_{i|SU(3)}(A,B;\xi_3) = K^{\underline{\alpha\beta}}_{SU(3)} + \frac{g_s\xi_3}{2}f^{\underline{\alpha\gamma\beta}}\left[\delta_{i2}\partial_\mu\left(\frac{\partial^\mu B^{\underline{\gamma}}}{\partial^2}\right) + \frac{\delta_{i1}}{4}B^{\underline{\gamma}}\right] \ , \text{ where } \left(\bar{K}^{\underline{\alpha\beta}}_i(\xi_3)\right)^\dagger = \bar{K}^{\underline{\alpha\beta}}_i(\xi_3) \ , \quad (6.1)$$

($K^{\hat{a}\hat{b}}_{SU(2)} = \partial^\mu D^{\hat{a}\hat{b}}_\mu$, $K^{\underline{\alpha\beta}}_{SU(3)} = \partial^\mu D^{\underline{\alpha\beta}}_\mu$), which are Hermitian with reference to the scalar products in the spaces of square-integrable functions $L_2(\mathbb{R}^{1,3})$ taking their values in the respective Lie algebras $su(N)$, $N = 2, 3$,

$$\left(f, \bar{K}_{i|SU(N)}g\right)_{(N)} = \left(g, \bar{K}^\dagger_{i|SU(N)}f\right)^*_{(N)} \ , \text{ where } \left(f, \bar{K}_{i|SU(N)}g\right)_{(N)} = \int d^4x\, d^4y f^m(x)\bar{K}^{mn}_{i|SU(N)}(x;y)g^n(y) \ , \quad (6.2)$$

with arbitrary test functions $f^m, g^n \in L_2(\mathbb{R}^{1,3})$. The eigenvalues $\lambda^n_{i|k}$, $n = (\hat{b}, \underline{\beta})$, $k = 0, 1, 2, \ldots$, in the equation $\bar{K}^{mn}_i(\boldsymbol{\xi})u^n_k = \lambda^n_{i|k}u^n_k$ are real-valued and are to determine the Gribov region $\Omega(\xi_2, \xi_3)$ as follows:

$$\Omega(\xi_2,\xi_3) \equiv \left\{A^{\hat{a}}_\mu, A^{\underline{\alpha}}_\mu : \partial^\mu\left(A^{\hat{a}}_\mu, A^{\underline{\alpha}}_\mu\right) = -\frac{1}{2}\left(\xi_2 B^{\hat{a}}, \xi_3 B^{\underline{\alpha}}\right), \left(K^{\hat{a}\hat{b}}_{SU(2)}, K^{\underline{\alpha\beta}}_{SU(3)}\right) > 0\right\} \ .$$

[15] The Hermitian extended operator $\bar{K}^{mn}_{1|SU(N)}(A,B;\xi)$ suggested in [64] was written with mistake.

The Hermitian operators $\bar{K}^{mn}_{i|SU(N)}(x)$ cannot be used equivalently, i.e., for any $i = 1, 2$, to determine the eigenvalues of non-Hermitian operator $K^{mn}_{SU(N)}(\xi_N)$. Indeed, a definition of the Gribov region requires that $K^{mn}_{SU(N)}(\xi_N)$ be positive definite. The case of $i = 2$ does satisfy this condition, and so we propose a form of the Gribov–Zwanziger functional in $R_{\boldsymbol{\xi}}$ -like gauges,

$$h^T(A, B; \xi_2, \xi_3) = h^T_{SU(2)}(A, B; \xi_2) + h^T_{SU(3)}(A, B; \xi_3) \ , \tag{6.3}$$

$$h^T_{SU(2)}(A, B; \xi_2) = \gamma_1^2(\boldsymbol{\xi}) g^2 \int d^4x \ d^4y \ \varepsilon^{\hat{a}\hat{b}\hat{c}} A^{\hat{b}T}_\mu(x) \left(\bar{K}^{-1}_{2|SU(2)}\right)^{\hat{a}\hat{d}}(x; y) \, \varepsilon^{\hat{d}\hat{e}\hat{c}} A^{\mu\hat{e}T}(y) + 4 \cdot 3g^2\gamma_1^2(\boldsymbol{\xi}) \ , \tag{6.4}$$

$$h^T_{SU(3)}(A, B; \xi_3) = \gamma_2^2(\boldsymbol{\xi}) g_s^2 \int d^4x \ d^4y \ f^{\underline{\alpha\beta\gamma}} A^{\underline{\beta}T}_\mu(x) \left(\bar{K}^{-1}_{2|SU(3)}\right)^{\underline{\alpha\delta}}(x; y) \, f^{\underline{\delta\sigma\gamma}} A^{\mu\underline{\sigma}T}(y) + 4 \cdot 8g_s^2\gamma_2^2(\boldsymbol{\xi}) \ . \tag{6.5}$$

which determines the Gribov region $\Omega(\boldsymbol{\xi})$. The thermodynamic Gribov parameters $\gamma_i^2(\boldsymbol{\xi})$ must depend on the gauge parameters $\boldsymbol{\xi}$ so as to be determined in a self-consistent way from the relations (5.35), (5.36), involving the functional $S_{F_{\mathbf{0}},h}(\phi)$ and the vacuum energy $\mathcal{E}_{\rm vac}(\boldsymbol{\xi})$. In fact, the parameters $\gamma_i^2(\boldsymbol{\xi})$, $i = 1, 2$, must depend on ξ_2, ξ_3, albeit with the horizon functional $h^T(A, B; \xi_2, \xi_3)$ in (6.3), instead of $h_{\mathbf{0}}$ given by the Landau gauge. The suggested introduction of the Gribov–Zwanziger horizon functional is based on a representation of the Yang–Mills connection by using the transverse, $A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu$, and longitudinal, $A^{\hat{a}\mathrm{L}}_\mu, A^{\underline{\alpha}\mathrm{L}}_\mu$, components:

$$\begin{aligned} \left(A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu\right) &= \left(\delta^\nu_\mu - \frac{\partial_\mu\partial^\nu}{\partial^2}\right)\left(A^{\hat{a}}_\nu, A^{\underline{\alpha}}_\nu\right) = \left(A^{\hat{a}}_\mu, A^{\underline{\alpha}}_\mu\right) + \frac{\partial_\mu}{2\partial^2}(\xi_2 B^{\hat{a}}, \xi_3 B^{\underline{\alpha}}) \ , \\ \left(A^{\hat{a}\mathrm{L}}_\mu, A^{\underline{\alpha}\mathrm{L}}_\mu\right) &= \frac{\partial_\mu\partial^\nu}{\partial^2}\left(A^{\hat{a}}_\nu, A^{\underline{\alpha}}_\nu\right) = -\frac{\partial_\mu}{2\partial^2}(\xi_2 B^{\hat{a}}, \xi_3 B^{\underline{\alpha}}) \ , \end{aligned} \tag{6.6}$$

so that the $R_{\boldsymbol{\xi}}$-like gauge induced by the gauge Boson $F_{\boldsymbol{\xi}}$ in (5.15) is equivalent to the conditions

$$\partial^\mu\left(A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu\right) = 0 \ , \quad \partial^\mu\left(A^{\hat{a}\mathrm{L}}_\mu, A^{\underline{\alpha}\mathrm{L}}_\mu\right) = -\frac{1}{2}(\xi_2 B^{\hat{a}}, \xi_3 B^{\underline{\alpha}}) \ .$$

As a consequence, the operators $\bar{K}^{\hat{a}\hat{b}}_{2|SU(2)}$ and $\bar{K}^{\underline{\alpha\beta}}_{2|SU(3)}$ are nothing else than the Faddeev–Popov operators for the transverse components of the gauge fields $\left(A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu\right)$, which determine the physical degrees of freedom,

$$\left(\bar{K}^{\hat{a}\hat{b}}_{2|SU(2)}, \bar{K}^{\underline{\alpha\beta}}_{2|SU(3)}\right) = \partial^\mu\left[\left(\partial_\mu\delta^{\hat{a}\hat{b}}, \partial_\mu\delta^{\underline{\alpha\beta}}\right) + \left(g\varepsilon^{\hat{a}\hat{c}\hat{b}}A^{\hat{c}\mathrm{T}}_\mu, g_s f^{\underline{\alpha\gamma\beta}}A^{\underline{\gamma}\mathrm{T}}_\mu\right)\right] = \left(K^{\hat{a}\hat{b}}_{SU(2)}, K^{\underline{\alpha\beta}}_{SU(3)}\right)(A^{\mathrm{T}}) \ . \tag{6.7}$$

To provide a justification of the horizon functional (6.3), (6.5), we examine the following

Proposition 7 *For the transverse components* $\left(A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu\right) \in \Omega(\xi_2, \xi_3)$ *of the gauge fields, the equations*

$$\left(K^{\hat{a}\hat{b}}_{SU(2)}(A)\varsigma^{\hat{b}}, K^{\underline{\alpha\beta}}_{SU(3)}(A)\varsigma^{\underline{\beta}}\right) = (0, 0) \tag{6.8}$$

for arbitrary field configurations $\left(A^{\hat{a}}_\mu, A^{\underline{\alpha}}_\mu\right)$ *admit only the vanishing solutions* $(\varsigma^{\hat{b}}, \varsigma^{\underline{\beta}}) = (0, 0)$ *in the class of functions regular in* ξ_2, ξ_3.

A proof is based on the hermiticity of $K^{\hat{a}\hat{b}}_{SU(2)}(A^{\mathrm{T}})$, $K^{\underline{\alpha\beta}}_{SU(3)}(A^{\mathrm{T}})$, due to the relations (6.1), (6.7), which implies their invertibility and positive definitiveness. The regularity imposed on the respective zero-mode parameters $\varsigma^{\hat{b}}(x, \xi_2) = \varsigma^{\hat{b}}(\xi_2)$, $\varsigma^{\underline{\beta}}(x, \xi_3) = \varsigma^{\underline{\beta}}(\xi_3)$ for the operators $K^{\hat{a}\hat{b}}_{SU(2)}(A)$, $K^{\underline{\alpha\beta}}_{SU(3)}$ implies the possibility of their representation as power series in the respective gauge parameters ξ_2, ξ_3,

$$\varsigma^{\hat{b}}(\xi_2) = \sum_{n\geq 0}\varsigma^{\hat{b}}_n(\xi_2)^n \ , \quad \varsigma^{\underline{\beta}}(\xi_3) = \sum_{n\geq 0}\varsigma^{\underline{\beta}}_n(\xi_3)^n \ , \tag{6.9}$$

which converge within certain convergence radiuses R_2, R_3, respectively. From (6.8) it follows that

$$\left(\varsigma^{\hat{b}}(\xi_2), \varsigma^{\underline{\beta}}(\xi_3)\right) = -\frac{1}{2}\left(g\xi_2\varepsilon^{\hat{a}\hat{c}\hat{d}}\left(\bar{K}^{-1}_{2|SU(2)}\right)^{\hat{b}\hat{a}}\partial_\mu\left(\frac{\partial^\mu B^{\hat{c}}}{\partial^2}\right)\varsigma^{\hat{d}}(\xi_2)\ ,\ g_s\xi_3 f^{\underline{\alpha\gamma\delta}}\left(\bar{K}^{-1}_{2|SU(3)}\right)^{\underline{\beta\alpha}}\partial_\mu\left(\frac{\partial^\mu B^{\underline{\gamma}}}{\partial^2}\right)\varsigma^{\underline{\delta}}(\xi_3)\right) \tag{6.10}$$

$$\stackrel{6.9}{\Longrightarrow}\left\{\begin{array}{l}\sum_{n\geq 0}\varsigma^{\hat{b}}_n(\xi_2)^n = -\frac{g}{2}\varepsilon^{\hat{a}\hat{c}\hat{d}}\sum_{n\geq 0}(\xi_2)^{n+1}\left(\bar{K}^{-1}_{2|SU(2)}\right)^{\hat{b}\hat{a}}\partial_\mu\left[\frac{\partial^\mu B^{\hat{c}}}{\partial^2}\right]\varsigma^{\hat{d}}_n = \sum_{n\geq 0}(\xi_2)^{n+1}\varphi^{\hat{b}}_n,\\ \sum_{n\geq 0}\varsigma^{\underline{\beta}}_n(\xi_3)^n = -\frac{g_s}{2}f^{\underline{\alpha\gamma\delta}}\sum_{n\geq 0}(\xi_3)^{n+1}\left(\bar{K}^{-1}_{2|SU(3)}\right)^{\underline{\beta\alpha}}\partial_\mu\left[\frac{\partial^\mu B^{\underline{\gamma}}}{\partial^2}\right]\varsigma^{\underline{\delta}}_n = \sum_{n\geq 0}(\xi_3)^{n+1}\varphi^{\underline{\beta}}_n\end{array}\right. . \tag{6.11}$$

The system of equations (6.11) for unknowns functions $\varsigma^{\hat{b}}_n$, $\varsigma^{\underline{\beta}}_n$ at a fixed order in n, starting from $n = 0$, yields the solution

$$\left(\varsigma^{\hat{b}}_0,\ \varsigma^{\underline{\beta}}_0\right) = (0,0) \stackrel{6.11}{\Longrightarrow} \left(\varphi^{\hat{b}}_0,\ \varphi^{\underline{\beta}}_0\right) = (0,0). \tag{6.12}$$

Therefore, $\left(\varsigma^{\hat{b}}_n,\ \varsigma^{\underline{\beta}}_n\right) = \left(\xi_2\varphi^{\hat{b}}_{n-1},\ \xi_3\varphi^{\underline{\beta}}_{n-1}\right)$ implies subsequently $\left(\varsigma^{\hat{b}}_n,\ \varsigma^{\underline{\beta}}_n\right) = 0$, $\left(\varsigma^{\hat{b}}_n,\ \varsigma^{\underline{\beta}}_n\right) = (0,0)$, for $n = 1,2,\ldots.$ As a result, the series (6.9) in their respective convergence regions R_2, R_3 vanish identically, which thereby proves the proposition.

Notice that the choice for the zero-modes of the respective Faddeev–Popov operators to be regular in ξ_2, ξ_3 is based on the assumption that we obtain the Gribov region for the Landau gauge in the limit $(\xi_2,\xi_3) \to (0,0)$. At the same time, Proposition 7 means that the Gribov region $\Omega(\xi_2,\xi_3)$ contains only the transverse components of the gauge fields:

$$\Omega(\xi_2,\xi_3) \equiv \left\{A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu : \partial^\mu\left(A^{\hat{a}\mathrm{T}}_\mu, A^{\underline{\alpha}\mathrm{T}}_\mu\right) = (0,0),\ \left(\bar{K}^{\hat{a}\hat{b}}_{2|SU(2)}, \bar{K}^{\underline{\alpha\beta}}_{2|SU(3)}\right) > 0\right\}\ .$$

We can thereby construct the Gribov–Zwanziger theory in arbitrary (covariant) $R_{\boldsymbol{\xi}}$-like gauges, suggested earlier [64, 94], as an extension of the BRST-invariant Faddeev–Popov action for the Yang–Mills theory in the case of a BRST-antiBRST-invariant quantum action for the Standard Model, in a way different from the one suggested by the equation (5.42) for $S_{F_{\boldsymbol{\xi}},h_{\boldsymbol{\xi}}}$. Our proposal has the form

$$S_{F_{\boldsymbol{\xi}},h^T_{\boldsymbol{\xi}}} = S_{\mathrm{SM}}\left(A\right) - \left(1/2\right)F_{\boldsymbol{\xi}}\overleftarrow{s}^2 + h^{\mathrm{T}}(A,B;\xi_2,\xi_3)\ . \tag{6.13}$$

There remains the question of establishing the coincidence of $Z_{\mathrm{SM},h_{\boldsymbol{\xi}}}(\eta)$ in (5.42) with $Z_{\mathrm{SM},h^T_{\boldsymbol{\xi}}}(\eta)$ determined using $h^{\mathrm{T}}(A,B;\xi_2,\xi_3)$, namely,

$$Z_{\mathrm{SM},h_{\boldsymbol{\xi}}}(\eta) \stackrel{?}{=} Z_{\mathrm{SM},h^{\mathrm{T}}_{\boldsymbol{\xi}}}(\eta)\ . \tag{6.14}$$

We intend to study this problem in separate research.

In addition, there are various lines of research for extending the results of the present work. First, the study of finite field-dependent BRST transformations in the multilevel formalism [95, 96] involving non-Abelian hypergauges and a non-trivial geometry. Second, the study of the Gribov ambiguity in generalized Hamiltonian formalism, as well as the study of a Hamiltonian Gribov–Zwanziger theory – see [88] – for Yang–Mills theories in the Lagrangian description using different gauges by means of finite field-dependent BRST(-antiBRST) transformations. Third, the study of an explicit relation between the two approaches using the finite field-dependent BRST transformations in the Yang–Mills theory [24, 29] and general gauge theories [31]. Fourth, the influence of renormalizability on the properties of the ingredients of BRST-antiBRST quantization at finite BRST-antiBRST transformations is also an open problem. We are, however, convinced that the presence of a gauge-invariant regularization which respects the Ward identities will replicate the properties of the non-renormalized theory by the properties of the renormalized one.

In the Standard Model, due to the presence of chiral Fermions in the lepton sector, described by the Lagrangian (5.4), one can adopt a gauge-invariant regularization as the higher derivative regularization [97, 98], which is the

Pauli–Villars regularization extended by higher-derivative terms. The first successful application of this regularization to the calculation of the one-loop effective action in the BRST-invariant Yang–Mills theory has been given by [99, 100, 101]. This regularization, when adapted to $N = 1$ supersymmetric field theory models [102, 103], preserves explicit supersymmetry, unlike the standard dimensional regularization, and has been recently elaborated in the $N = 2$ supersymmetric Yang–Mills theory interacting with matter [104], thereby respecting gauge invariance and $N = 2$ supersymmetry. In its turn, the dimensional regularization has been recently used [105] to study the problem of gauge-dependence in terms of the Ward identities, including the case of beta-functions, for renormalizable and non-renormalizable general chiral gauge theories in the BV quantization method. This regularization can also be implemented, but only in those parts of the Standard Model which do not include the lepton fields. The dimensional regularization has been partially applied [106] to the electroweak sector described by the Lagrangian (5.2). This is done using the method of algebraic renormalization [107] and aiming to describe electroweak interactions in the Standard Model to all orders of perturbation theory under BRST symmetry, with the infrared-finiteness of the off-shell Green functions, however, without the fulfilment of the Gribov "no-pole condition" [44] for the ghost Green functions. Therefore, a mathematically rigorous renormalization of the Standard Model in BRST and BRST-antiBRST quantization remains a topical problem.

Let us finally mention the search for an equivalent local description of the Gribov horizon functional by using a set of auxiliary fields, as in [46], such that it should be consistent with both the infinitesimal and finite forms of BRST-antiBRST invariance.

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Appendix

A Linearized Transformations

In this appendix, we make an explicit calculation of the Jacobian corresponding to linearized finite BRST-antiBRST transformations, i.e., transformations corresponding to the part of finite BRST-antiBRST transformations being linear in parameters of a special form, $\lambda_a = s_a \Lambda$. To this end, notice that, in virtue of (3.3)–(3.8), the quantity $\Im$ can be subsequently transformed as follows:

$$
\begin{aligned}
\Im &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\left(M^n\right) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\left(P+Q\right)^n = -\sum_{n=1}^{3} \frac{(-1)^n}{n} \mathrm{Str}\left(P+Q\right)^n - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\left(P+Q\right)^n \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\left(P^n\right) - \sum_{n=2}^{\infty} (-1)^n \mathrm{Str}\left(P^{n-1}Q\right) - \sum_{n=2}^{3} \frac{(-1)^n}{n} C_n^2 \mathrm{Str}\left(P^{n-2}Q^2\right) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} K_n \mathrm{Str}\left(P^{n-3}QPQ\right) \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\left(P^n\right) - \frac{1}{2}\mathrm{Str}\left(Q^2\right) + \mathrm{Str}\left(PQ^2\right) + \sum_{n=1}^{\infty} (-1)^n \mathrm{Str}\left(P^nQ\right) + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n n \mathrm{Str}\left(P^nQPQ\right) \ . \qquad \text{(A.1)}
\end{aligned}
$$

whence

$$\Im = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}f^{n-1}\mathrm{Str}\,(P) - \frac{1}{2}\mathrm{Str}\left(Q^2\right) + \mathrm{Str}\,(QPQ) + \sum_{n=1}^{\infty}(-1)^n\,f^{n-1}\mathrm{Str}\,(QP) + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\,nf^{n-1}\mathrm{Str}\,(QPQP)\ . \tag{A.2}$$

Finally,

$$\begin{aligned}\Im = {} & 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}f^n - \mathrm{Str}\,(R) + (1+f)\,\mathrm{Str}\,(Q_2Q) + \mathrm{Str}\,(Q_2)\sum_{n=1}^{\infty}(-1)^n\,f^{n-1}\,(1+f) \\ & + \frac{1}{2}\mathrm{Str}\left(Q_2^2\right)\sum_{n=1}^{\infty}(-1)^n\,nf^{n-1}\,(1+f)^2 \equiv -2\ln(1+f) + \Re\ ,\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}&\Re = -\mathrm{Str}\,(R) + (1+f)\cdot\mathrm{Str}\,(Q_2Q) + \varphi(f)\cdot\mathrm{Str}\,(Q_2) + \frac{1}{2}\psi(f)\cdot\mathrm{Str}\left(Q_2^2\right)\ , \\ &\varphi(x) = (1+x)\sum_{n=1}^{\infty}(-1)^n\,x^{n-1}\ , \quad \psi(x) = (1+x)^2\sum_{n=1}^{\infty}(-1)^n\,nx^{n-1}\ .\end{aligned} \tag{A.4}$$

Let us study the formal series $\varphi(x)$ and $\psi(x)$:

$$\begin{aligned}\varphi(x) &= (1+x)\sum_{n=1}^{\infty}(-1)^n\,x^{n-1} = -(1+x)\sum_{m=0}^{\infty}(-1)^m\,x^m = -(1+x)\frac{1}{(1+x)} = -1\ , \\ \psi(x) &= (1+x)^2\sum_{n=1}^{\infty}(-1)^n\,nx^{n-1} = (1+x)^2\frac{\partial}{\partial x}\sum_{n=1}^{\infty}(-1)^n\,x^n = (1+x)^2\frac{\partial}{\partial x}\left(\frac{1}{1+x}-1\right) \\ &= (1+x)^2\frac{\partial}{\partial x}\left(\frac{1}{1+x}\right) = -(1+x)^2\frac{1}{(1+x)^2} = -1\ ,\end{aligned} \tag{A.5}$$

Therefore,

$$\Re = -\mathrm{Str}\,(R) + (1+f)\,\mathrm{Str}\,(Q_2Q) - \mathrm{Str}\,(Q_2) - \frac{1}{2}\mathrm{Str}\left(Q_2^2\right) = -\mathrm{Str}\left[R + Q_2 + \frac{1}{2}Q_2^2 - (1+f)\,(Q_2Q)\right]\ ,$$

which proves the relation (3.9).

B Transformations with Arbitrary Parameters

In this appendix, we prove Lemmas 1–5 and present explicit calculations related to the Jacobian of finite BRST-antiBRST transformations with arbitrary field-dependent parameters λ_a.

B.1 Proof of Lemma 1

Considering the relations (4.4), we examine the quantities $\mathrm{Str}\left(P^{n-1}R\right)$ which obey

$$\mathrm{Str}\left(P^{n-1}R\right) = \begin{cases}\mathrm{Str}\,(R)\ , & n=1\ , \\ 0\ , & n>1\ .\end{cases} \tag{B.1}$$

Indeed, due to $Y^A_{,B}X^{Bb}=0$, we can write down a chain of relations:

$$
\begin{aligned}
&(RP)_B^A = R_D^A P_B^D = -\frac{1}{2}\lambda^2 \left(\frac{\delta Y^A}{\delta \phi^D} X^{Db}\right) \frac{\delta \lambda_b}{\delta \phi^B} = 0 \ , \\
&\left(RP^2\right)_B^A = (RP)_D^A P_B^D = 0 \ , \\
&\ldots \\
&\left(RP^{n-1}\right)_B^A = (RP)_D^A \left(P^{n-1}\right)_B^D = 0 \ , \quad n > 1 \ .
\end{aligned} \tag{B.2}
$$

Using the property $\mathrm{Str}\,(AB) = \mathrm{Str}\,(BA)$ for even matrices,

$$
\mathrm{Str}\left(P^{n-1}R\right) = \mathrm{Str}\left(RP^{n-1}\right) = 0 \ , \tag{B.3}
$$

we arrive at

$$
\mathrm{Str}\,(M^n) = \mathrm{Str}\,(P+Q)^n + n\mathrm{Str}\left(P^{n-1}R\right) = \begin{cases} \mathrm{Str}\,(P+Q) + \mathrm{Str}\,(R) \ , & n = 1 \ , \\ \mathrm{Str}\,(P+Q)^n \ , & n > 1 \ , \end{cases}
$$

which thereby proves Lemma 1.

B.2 Proof of Lemma 2

Considering the contribution $\mathrm{Str}\,(P+Q)^n$ in (4.5), we notice that an occurrence of $Q \sim \lambda_a$ more then twice yields zero, $\lambda_a \lambda_b \lambda_c \equiv 0$. A direct calculation for $n = 2, 3$ leads to the binomial rule

$$
\mathrm{Str}\,(P+Q)^n = \sum_{k=0}^{n} C_n^k \mathrm{Str}\left(P^{n-k}Q^k\right) = \mathrm{Str}\left(P^n + nP^{n-1}Q + C_n^2 P^{n-2}Q^2\right) \ , \tag{B.4}
$$

whereas the case $n = 4$ fails to conform to this rule due to the presence of the products $PQPQ$ and $QPQP$, which cannot be rearranged to the form Q^2P^2 under the symbol of supertrace by using the property $\mathrm{Str}\,(AB) = \mathrm{Str}\,(BA)$. On the other hand, this property allows one to present the case $n = 4$ as follows:

$$
\mathrm{Str}\,(P+Q)^4 = \mathrm{Str}\left(P^4 + 4P^3Q + 4P^2Q^2 + 2PQPQ\right) \ . \tag{B.5}
$$

The consideration of the case $n > 4$ is simplified by the fact that one needs to keep track of the products that contain the matrix Q no more than twice, i.e., we only need to retain P^n, $P^{n-1}Q$ and pairs of Q's, while separating the expressions reduced to $P^{n-2}Q^2$ from those containing pairs of Q's so "sandwiched" between P's as not to allow their rearrangement into $P^{n-2}Q^2$ by using the property (3.3). Starting from the case $n = 4$, given by (B.5), and considering a monomial p^2q^2 composed by c-numbers p, q, we find that under the symbol of supertrace the coefficient C_4^2 decomposes into C_4^1 for P^2Q^2 and C_2^1 for $(PQ)^2$, $C_4^2 = C_4^1 + C_2^1$, so that

$$
\mathrm{Str}\,(P+Q)^4 = \sum_{k=0}^{1} C_4^k \mathrm{Str}\left(P^{4-k}Q^k\right) + C_4^1 \mathrm{Str}\left(P^2Q^2\right) + C_2^1 \mathrm{Str}\,(PQPQ) \ . \tag{B.6}
$$

For $n = 5$, we consider a c-number monomial p^3q^2 and find that the coefficient C_5^2 decomposes into C_5^1 for P^3Q^2 and C_5^1 for $P(PQ)^2$, $C_5^2 = C_5^1 + C_5^1$, so that

$$
\mathrm{Str}\,(P+Q)^5 = \sum_{k=0}^{1} C_5^k \mathrm{Str}\left(P^{5-k}Q^k\right) + C_5^1 \mathrm{Str}\left(P^3Q^2\right) + C_5^1 \mathrm{Str}\left[P\,(PQ)^2\right] \ . \tag{B.7}
$$

For $n = 6$, we consider a c-number monomial p^4q^2 and find that the coefficient C_6^2 decomposes into C_6^1 for P^4Q^2, C_6^1 for $P^2(PQ)^2$, and C_3^1 for $(P^2Q)^2$, $C_6^2 = C_6^1 + C_6^1 + C_3^1$, so that

$$
\mathrm{Str}\,(P+Q)^6 = \sum_{k=0}^{1} C_6^k \mathrm{Str}\left(P^{6-k}Q^k\right) + C_6^1 \mathrm{Str}\left(P^4Q^2\right) + C_6^1 \mathrm{Str}\left[P^2\,(PQ)^2\right] + C_3^1 \mathrm{Str}\left[\left(P^2Q\right)^2\right] \ . \tag{B.8}
$$

For $n=7$, we consider a c-number monomial p^5q^2 and find that the coefficient C_7^2 decomposes into C_7^1 for P^5Q^2, C_7^1 for $P^3(PQ)^2$, and C_7^1 for $P(P^2Q)^2$, $C_7^2 = C_7^1 + C_7^1 + C_7^1$, so that

$$\begin{aligned}\mathrm{Str}\,(P+Q)^7 &= \sum_{k=0}^{1} C_7^k \mathrm{Str}\left(P^{7-k}Q^k\right) + C_7^1 \mathrm{Str}\left(P^5Q^2\right) \\ &+ C_7^1 \mathrm{Str}\left[P^3\left(PQ\right)^2\right] + C_7^1 \mathrm{Str}\left[P\left(P^2Q\right)^2\right] \ .\end{aligned}$$

For $n=8$, we consider a c-number monomial p^6q^2 and find that the coefficient C_8^2 decomposes into C_8^1 for P^6Q^2, C_8^1 for $P^4(PQ)^2$, C_8^1 for $P^2(P^2Q)^2$, and C_4^1 for $(P^3Q)^2$, $C_8^2 = 3C_8^1 + C_4^1$, so that

$$\begin{aligned}\mathrm{Str}\,(P+Q)^8 &= \sum_{k=0}^{1} C_8^k \mathrm{Str}\left(P^{8-k}Q^k\right) + C_8^1 \mathrm{Str}\left(P^6Q^2\right) \\ &+ C_8^1 \mathrm{Str}\left[P^4\left(PQ\right)^2\right] + C_8^1 \mathrm{Str}\left[P^2\left(P^2Q\right)^2\right] + C_4^1 \mathrm{Str}\left[\left(P^3Q\right)^2\right] .\end{aligned}$$

Proceeding by induction for $n=2k$ and considering c-number monomials $p^{2(k-1)}q^2$, we find that the coefficient C_{2k}^2 decomposes into C_{2k}^1 for $P^{2(k-1)}Q^2$, C_{2k}^1 for $P^{2(k-2)}(PQ)^2$, C_{2k}^1 for $P^{2(k-3)}(P^2Q)^2$,…,C_{2k}^1 for $P^2(P^{(k-2)}Q)^2$, and C_k^1 for $(P^{(k-1)}Q)^2$, $C_{2k}^2 = (k-1)C_{2k}^1 + C_k^1$, so that

$$\begin{aligned}\mathrm{Str}\,(P+Q)^{2k} &= \sum_{l=0}^{1} C_{2k}^l \mathrm{Str}\left(P^{2k-l}Q^l\right) + C_{2k}^1 \sum_{l=0}^{k-2} \mathrm{Str}\left[P^{2(k-l-1)}\left(P^lQ\right)^2\right] \\ &+ C_k^1 \mathrm{Str}\left[\left(P^{k-1}Q\right)^2\right] , \quad k \geq 2 \ .\end{aligned} \tag{B.9}$$

For $n=2k+1$, we consider c-number monomials $p^{2k-1}q^2$ and find that the coefficient C_{2k+1}^2 decomposes into C_{2k+1}^1 for $P^{2k-1}Q^2$, C_{2k+1}^1 for $P^{2k-3}(PQ)^2$, C_{2k+1}^1 for $P^{2k-5}(P^2Q)^2$,…, C_{2k+1}^1 for $P^3(P^{(k-2)}Q)^2$, and C_{2k+1}^1 for $P(P^{(k-1)}Q)^2$, $C_{2k+1}^2 = kC_{2k+1}^1$, so that

$$\mathrm{Str}\,(P+Q)^{2k+1} = \sum_{l=0}^{1} C_{2k+1}^l \mathrm{Str}\left(P^{2k+1-l}Q^l\right) + C_{2k+1}^1 \sum_{l=0}^{k-1} \mathrm{Str}\left[P^{2(k-l)-1}\left(P^lQ\right)^2\right] \ , \quad k \geq 2 \ . \tag{B.10}$$

Formulae (B.9), (B.10) thereby prove Lemma 2.

B.3 Proof of Lemma 3

Due to the relations $X^{Aa}_{,A} = 0$, we have

$$\mathrm{Str}\,(Q_1) = (Q_1)^A_A (-1)^{\varepsilon_A} = \frac{\delta X^{Aa}}{\delta\phi^A}\lambda_a = 0 \ . \tag{B.11}$$

We also observe the following:

$$\mathrm{Str}\left(Q_1^2\right) = \left(Q_1^2\right)^A_A (-1)^{\varepsilon_A} = \frac{\delta X^{Aa}}{\delta\phi^B}\lambda_a \frac{\delta X^{Bb}}{\delta\phi^A}\lambda_b (-1)^{\varepsilon_B} = \frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^A}\lambda_b\lambda_a (-1)^{\varepsilon_A} \ . \tag{B.12}$$

Differentiating the relation $X^{Aa}_{,B}X^{Bb} = \varepsilon^{ab}Y^A$ with respect to ϕ^A, we find

$$\frac{\delta}{\delta\phi^B}\left(\frac{\delta X^{Aa}}{\delta\phi^A}\right) X^{Bb} (-1)^{\varepsilon_B} + \frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^A} + \varepsilon^{ba}\frac{\delta Y^A}{\delta\phi^A} = 0 \ . \tag{B.13}$$

Once again, using the relation $X^{Aa}_{,A} = 0$, we have

$$\frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^A} = \varepsilon^{ab}\frac{\delta Y^A}{\delta\phi^A} \ , \tag{B.14}$$

whence

$$\mathrm{Str}\left(Q_1^2\right) = \varepsilon^{ab}\frac{\delta Y^A}{\delta\phi^A}\lambda_b\lambda_a\left(-1\right)^{\varepsilon_A} = -\frac{\delta Y^A}{\delta\phi^A}\lambda^2\left(-1\right)^{\varepsilon_A} = 2\mathrm{Str}\left(R\right)\ . \tag{B.15}$$

Relations (B.11) and (B.15) thereby prove Lemma 3.

B.4 Proof of Lemma 4

Let us write down a chain of relations:

$$\begin{aligned}
&P^A_B = X^{Aa}\frac{\delta\lambda_a}{\delta\phi^A}\ ,\\
&\left(P^2\right)^A_B = P^A_D P^D_B = X^{Aa}\left(\frac{\delta\lambda_a}{\delta\phi^B}X^{Bb}\right)\frac{\delta\lambda_b}{\delta\phi^A} = X^{Aa}m^b_a\frac{\delta\lambda_b}{\delta\phi^B}\ ,\\
&\left(P^3\right)^A_B = \left(P^2\right)^A_D P^D_B = X^{Aa}m^b_a\left(\frac{\delta\lambda_b}{\delta\phi^D}X^{Dd}\right)\frac{\delta\lambda_d}{\delta\phi^B} = X^{Aa}\left(m^2\right)^b_a\frac{\delta\lambda_b}{\delta\phi^B}\ ,\\
&\ldots\\
&\left(P^n\right)^A_B = \left(P^{n-1}\right)^A_D P^D_B = X^{Aa}\left(m^{n-2}\right)^b_a\left(\frac{\delta\lambda_b}{\delta\phi^D}X^{Dd}\right)\frac{\delta\lambda_d}{\delta\phi^B} = X^{Aa}\left(m^{n-1}\right)^b_a\frac{\delta\lambda_b}{\delta\phi^B}\ ,
\end{aligned} \tag{B.16}$$

whence

$$\mathrm{Str}\left(P^n\right) = \left(P^n\right)^A_A\left(-1\right)^{\varepsilon_A} = -\left(m^{n-1}\right)^b_a\left(\frac{\delta\lambda_b}{\delta\phi^A}X^{Aa}\right) = -\left(m^{n-1}\right)^b_a m^a_b = -\left(m^n\right)^a_a\ , \tag{B.17}$$

which thereby proves Lemma 4.

B.5 Proof of Lemma 5

Let us consider the matrix

$$\begin{aligned}
(QP)^A_B \equiv Q^A_D P^D_B &= \left(-1\right)^{\varepsilon_A+1}\lambda_a\left(\frac{\delta X^{Aa}}{\delta\phi^D} + Y^A\frac{\delta\lambda^a}{\delta\phi^D}\right)X^{Dd}\frac{\delta\lambda_d}{\delta\phi^B}\\
&= \left(-1\right)^{\varepsilon_A+1}\lambda_a\left[\varepsilon^{ab}Y^A + Y^A\left(s^b\lambda^a\right)\right]\frac{\delta\lambda_b}{\delta\phi^B} \equiv \left(Q_2\right)^A_B + \left(-1\right)^{\varepsilon_A+1}m^{ba}\lambda_a Y^A\frac{\delta\lambda_b}{\delta\phi^B}\ .
\end{aligned} \tag{B.18}$$

Since in the case of arbitrary λ_a there is no information on the symmetry properties of $m^{ab} = s^a\lambda^b$, we thus arrive at a new matrix:

$$(Q_2^{(1)})^A_B \equiv \left(-1\right)^{\varepsilon_A+1}m^{ba}\lambda_a Y^A\frac{\delta\lambda_b}{\delta\phi^B}\ , \tag{B.19}$$

which is not contained among the matrices P, Q. If we now consider the matrix $Q_2 + Q_2^{(1)}$ acting on P,

$$\begin{aligned}
(Q_2 + Q_2^{(1)})^A_D P^D_B &= \left(-1\right)^{\varepsilon_A+1}\lambda_a\left(Y^A\frac{\delta\lambda^a}{\delta\phi^D} + m^{ba}Y^A\frac{\delta\lambda_b}{\delta\phi^D}\right)X^{Dd}\frac{\delta\lambda_d}{\delta\phi^B}\\
&= \left(-1\right)^{\varepsilon_A+1}\lambda_a\left[Y^A\left(s^d\lambda^a\right) + m^{ba}Y^A\left(s^d\lambda_b\right)\right]\frac{\delta\lambda_d}{\delta\phi^B}\\
&= \left(-1\right)^{\varepsilon_A+1}m^{da}\lambda_a Y^A\frac{\delta\lambda_d}{\delta\phi^B} + \left(-1\right)^{\varepsilon_A+1}m^d_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_d}{\delta\phi^B}\ ,
\end{aligned} \tag{B.20}$$

it follows that,

$$(Q_2 + Q_2^{(1)})^A_D P^D_B = (Q_2^{(1)})^A_B + \left(-1\right)^{\varepsilon_A+1}m^d_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_d}{\delta\phi^B}\ , \tag{B.21}$$

so we have another new matrix:

$$(Q_2^{(2)})^A_B \equiv \left(-1\right)^{\varepsilon_A+1}m^d_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_d}{\delta\phi^B}\ . \tag{B.22}$$

Let us, once again, consider a similar construction:

$$\begin{aligned}(Q_2^{(1)}+Q_2^{(2)})^A_D P^D_B &= \left[(-1)^{\varepsilon_A+1}\, m^{ba}\lambda_a Y^A\frac{\delta\lambda_b}{\delta\phi^D}+(-1)^{\varepsilon_A+1}\, m^c_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_c}{\delta\phi^D}\right]X^{Dd}\frac{\delta\lambda_d}{\delta\phi^B}\\ &= (-1)^{\varepsilon_A+1}\, m^{ba}\lambda_a Y^A\frac{\delta\lambda_b}{\delta\phi^D}X^{Dd}\frac{\delta\lambda_d}{\delta\phi^B}+(-1)^{\varepsilon_A+1}\, m^c_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_c}{\delta\phi^D}X^{Dd}\frac{\delta\lambda_d}{\delta\phi^B}\\ &= (-1)^{\varepsilon_A+1}\, m^d_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_d}{\delta\phi^B}+(-1)^{\varepsilon_A+1}\, m^d_c m^c_b m^{ba}\lambda_a Y^A\frac{\delta\lambda_d}{\delta\phi^B}\equiv(Q_2^{(2)}+Q_2^{(3)})^A_B\ . \qquad \text{(B.23)}\end{aligned}$$

Generally, the above process leads to

$$(Q_2^{(n)}+Q_2^{(n+1)})P = Q_2^{(n+1)}+Q_2^{(n+2)}\ ,\quad n\geq 0\ ,\quad Q_2^{(0)}\equiv Q_2\ , \qquad \text{(B.24)}$$

so that there emerges an infinite sequence of objects $Q_2^{(n)}$ constructed by multiplication of the matrix with the elements $m^a_b = s^a\lambda_b$. Using this observation and the fact that $m^{ba}\lambda_a = \left(s^b\lambda^a\right)\lambda_a = -\left(s^b\lambda_a\right)\lambda^a = -m^b_a\lambda^a$, let us rewrite the above relations containing m^{ab} in terms of m^a_b:

$$\begin{aligned}&(QP)^A_B = (Q_2)^A_B + (-1)^{\varepsilon_A}\, m^b_a\lambda^a Y^A\frac{\delta\lambda_b}{\delta\phi^B} = (Q_2+Q_2^{(1)})^A_B\ ,\\ &(Q_2+Q_2^{(1)})^A_B P^D_B = (-1)^{\varepsilon_A}\, m^b_a\lambda^a Y^A\frac{\delta\lambda_b}{\delta\phi^B}+(-1)^{\varepsilon_A}\, m^d_b m^b_a\lambda^a Y^A\frac{\delta\lambda_d}{\delta\phi^B} = (Q_2^{(1)}+Q_2^{(2)})^A_B\ ,\\ &(Q_2^{(1)}+Q_2^{(2)})^A_B P^D_B = (-1)^{\varepsilon_A}\, m^d_b m^b_a\lambda^a Y^A\frac{\delta\lambda_d}{\delta\phi^B}+(-1)^{\varepsilon_A}\, m^d_c m^c_b m^b_a\lambda^a Y^A\frac{\delta\lambda_d}{\delta\phi^B} = (Q_2^{(2)}+Q_2^{(3)})^A_B\ ,\\ &\ldots\\ &(Q_2^{(n)}+Q_2^{(n+1)})^A_D P^D_B = (-1)^{\varepsilon_A}\left(m^{n+1}\right)^b_a\lambda^a Y^A\frac{\delta\lambda_b}{\delta\phi^B}+(-1)^{\varepsilon_A}\left(m^{n+2}\right)^b_a\lambda^a Y^A\frac{\delta\lambda_b}{\delta\phi^B} = (Q_2^{(n+1)}+Q_2^{(n+2)})^A_B\ ,\ n\geq 0\ ,\end{aligned} \qquad \text{(B.25)}$$

which implies

$$(Q_2^{(n)})^A_B = (-1)^{\varepsilon_A}\,(m^n)^a_b\lambda^b Y^A\frac{\delta\lambda_a}{\delta\phi^B}\ ,\quad n\geq 0\ , \qquad \text{(B.26)}$$

Using the matrix $Y = (Y^b_a)^A_B$ given by

$$(Y^b_a)^A_B \equiv (-1)^{\varepsilon_A}\,\lambda^b Y^A\frac{\delta\lambda_a}{\delta\phi^B}\ ,\quad (Y^a_a)^A_B = (Q_2)^A_A\ ,\quad \mathrm{tr}\,(Y) = Q_2\ ,$$

we can represent the above sequence as follows:

$$Q_2^{(n)} = \mathrm{tr}\,(m^n Y)\ ,\quad n\geq 0\ , \qquad \text{(B.27)}$$

Hence, taking account of the property (B.24), we have

$$\begin{aligned}&(Q_1+Q_2)\,P = Q_2+Q_2^{(1)} = \mathrm{tr}\,(Y+mY) = \mathrm{tr}\,[(e+m)\,Y]\ ,\\ &(Q_1+Q_2)\,P^2 = Q_2^{(1)}+Q_2^{(2)} = \mathrm{tr}\left(mY+m^2Y\right) = \mathrm{tr}\,[m\,(e+m)\,Y]\ ,\\ &(Q_1+Q_2)\,P^3 = Q_2^{(2)}+Q_2^{(3)} = \mathrm{tr}\left(m^2Y+m^3Y\right) = \mathrm{tr}\left[m^2\,(e+m)\,Y\right]\ ,\\ &\ldots\\ &(Q_1+Q_2)\,P^n = Q_2^{(n-1)}+Q_2^{(n)} = \mathrm{tr}\left(m^{n-1}Y+m^nY\right) = \mathrm{tr}\left[m^{n-1}\,(e+m)\,Y\right]\ ,\quad n\geq 1\ .\end{aligned} \qquad \text{(B.28)}$$

Recalling that $Q = Q_1+Q_2$, we finally have

$$QP^n = \mathrm{tr}\left[m^{n-1}\,(e+m)\,Y\right]\ ,\quad n\geq 1\ ,$$

which completes the proof of Lemma 5.

B.6 Calculation of Jacobian

Let us consider a calculation of the quantity $\Im$ on the basis of the relations (4.3)–(4.14). First of all, we have

$$\Im = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(M^n\right) = \mathrm{Str}\left(R\right) - \sum_{n=1}^{3}\frac{(-1)^n}{n}\mathrm{Str}\left(P+Q\right)^n - \sum_{n=4}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(P+Q\right)^n\ . \tag{B.29}$$

Decomposing the summation number $n\geq 4$ into odd and even components, $n=(2k+1,2k)$, $k\geq 2$, we have, according to (4.7), (4.8),

$$\begin{aligned}
\Im &= \mathrm{Str}\left(R\right) - \sum_{n=1}^{3}\frac{(-1)^n}{n}\mathrm{Str}\left(P+Q\right)^n \\
&- \sum_{k=2}^{\infty}\frac{(-1)^{2k+1}}{2k+1}\left\{\sum_{l=0}^{1}C_{2k+1}^{l}\mathrm{Str}\left(P^{2k+1-l}Q^{l}\right) + C_{2k+1}^{1}\sum_{l=0}^{k-1}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right]\right\} \\
&- \sum_{k=2}^{\infty}\frac{(-1)^{2k}}{2k}\left\{\sum_{l=0}^{1}C_{2k}^{l}\mathrm{Str}\left(P^{2k-l}Q^{l}\right) + C_{2k}^{1}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(P^{l}Q\right)^2\right] + C_{k}^{1}\mathrm{Str}\left[\left(P^{k-1}Q\right)^2\right]\right\}\ ,
\end{aligned} \tag{B.30}$$

whence

$$\begin{aligned}
\Im &= \mathrm{Str}\left(R\right) - \sum_{n=1}^{3}\frac{(-1)^n}{n}\mathrm{Str}\left(P+Q\right)^n - \sum_{n=4}^{\infty}\frac{(-1)^n}{n}\sum_{l=0}^{1}C_{n}^{l}\mathrm{Str}\left(P^{n-l}Q^{l}\right) - \frac{1}{2}\sum_{k=2}^{\infty}\mathrm{Str}\left[\left(P^{k-1}Q\right)^2\right] \\
&+ \sum_{k=2}^{\infty}\sum_{l=0}^{k-1}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right] - \sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(P^{l}Q\right)^2\right]\ .
\end{aligned} \tag{B.31}$$

It should be noted that

$$\sum_{k=2}^{\infty}\sum_{l=0}^{k-1}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right] = \sum_{k=2}^{\infty}\mathrm{Str}\left[P\left(P^{k-1}Q\right)^2\right] + \sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right]\ , \tag{B.32}$$

whereas

$$\begin{aligned}
&\sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right] = \mathrm{Str}\left[P^{2(2-0)-1}\left(P^{0}Q\right)^2\right] + \sum_{k=3}^{\infty}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right] \\
&= \mathrm{Str}\left(P^3Q^2\right) + \sum_{k=3}^{\infty}\left\{\mathrm{Str}\left[P^{2(k-0)-1}\left(P^{0}Q\right)^2\right] + \sum_{l=1}^{k-2}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right]\right\} \\
&= \sum_{k=2}^{\infty}\mathrm{Str}\left(P^{2k-1}Q^2\right) + \sum_{k=3}^{\infty}\sum_{l=1}^{k-2}\mathrm{Str}\left[P^{2(k-l)-1}\left(P^{l}Q\right)^2\right]\ .
\end{aligned} \tag{B.33}$$

It should also be noted that

$$\begin{aligned}
&-\sum_{k=2}^{\infty}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(P^{l}Q\right)^2\right] = -\mathrm{Str}\left[P^{2(2-0-1)}\left(P^{0}Q\right)^2\right] - \sum_{k=3}^{\infty}\sum_{l=0}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(P^{l}Q\right)^2\right] \\
&= -\mathrm{Str}\left(P^2Q^2\right) - \sum_{k=3}^{\infty}\left\{\mathrm{Str}\left[P^{2(k-0-1)}\left(P^{0}Q\right)^2\right] + \sum_{l=1}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(P^{l}Q\right)^2\right]\right\} \\
&= -\sum_{k=2}^{\infty}\mathrm{Str}\left(P^{2(k-1)}Q^2\right) - \sum_{k=3}^{\infty}\sum_{l=1}^{k-2}\mathrm{Str}\left[P^{2(k-l-1)}\left(P^{l}Q\right)^2\right]\ .
\end{aligned} \tag{B.34}$$

From (B.31), (B.32), (B.33), (B.34), it follows that

$$
\begin{aligned}
\Im = {} & \mathrm{Str}\,(R) - \sum_{n=1}^{3} \frac{(-1)^n}{n} \mathrm{Str}\,(P+Q)^n - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \sum_{l=0}^{1} C_n^l \mathrm{Str}\left(P^{n-l} Q^l\right) \\
& + \sum_{k=2}^{\infty} \mathrm{Str}\left[P\left(P^{k-1}Q\right)^2 - \frac{1}{2}\left(P^{k-1}Q\right)^2\right] + \sum_{k=2}^{\infty} \mathrm{Str}\left[\left(P^{2k-1} - P^{2(k-1)}\right) Q^2\right] \\
& + \sum_{k=3}^{\infty} \sum_{l=1}^{k-2} \mathrm{Str}\left[\left(P^{2(k-l)-1} - P^{2(k-l-1)}\right)\left(P^l Q\right)^2\right] \ .
\end{aligned}
\tag{B.35}
$$

By virtue of (4.6),

$$
\begin{aligned}
& -\sum_{n=1}^{3} \frac{(-1)^n}{n} \mathrm{Str}\,(P+Q)^n = \mathrm{Str}\,(P+Q) - \sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\,(P+Q)^n \ , \\
& -\sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\,(P+Q)^n = -\sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\left(P^n + nP^{n-1}Q + C_n^2 P^{n-2} Q^2\right) \ , \quad n = 2, 3 \ ,
\end{aligned}
\tag{B.36}
$$

we have

$$
-\sum_{n=1}^{3} \frac{(-1)^n}{n} \mathrm{Str}\,(P+Q)^n = \mathrm{Str}\,(P+Q) - \sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\left(P^n + C_n^1 P^{n-1} Q\right) - \sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\left(C_n^2 P^{n-2} Q^2\right) \ , \tag{B.37}
$$

which implies

$$
\begin{aligned}
& -\sum_{n=1}^{3} \frac{(-1)^n}{n} \mathrm{Str}\,(P+Q)^n - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \sum_{l=0}^{1} C_n^l \mathrm{Str}\left(P^{n-l} Q^l\right) \\
& = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\,(P^n) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\left(C_n^1 P^{n-1} Q\right) - \sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\left(C_n^2 P^{n-2} Q^2\right) \ .
\end{aligned}
\tag{B.38}
$$

Consequently, using (B.35), we arrive at the representation

$$
\Im = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\,(P)^n + \Re \ , \tag{B.39}
$$

where, in virtue of the obvious relations

$$
\begin{aligned}
& \sum_{n=2}^{3} \frac{(-1)^n}{n} \mathrm{Str}\left(C_n^2 P^{n-2} Q^2\right) = \frac{1}{2} \mathrm{Str}\left(Q^2\right) - \mathrm{Str}\,(QPQ) \ , \\
& \sum_{n=1}^{\infty} (-1)^n \,\mathrm{Str}\left(P^{n-1} Q\right) = -\mathrm{Str}\,(Q) + \sum_{n=2}^{\infty} (-1)^n \,\mathrm{Str}\left(QP^{n-1}\right) \ ,
\end{aligned}
\tag{B.40}
$$

and, due to the property $\mathrm{Str}\,(AB) = \mathrm{Str}\,(BA)$, we have

$$
\begin{aligned}
\Re = {} & \mathrm{Str}\,(R) - \frac{1}{2} \mathrm{Str}\left(Q^2\right) + \mathrm{Str}\,(QPQ) + \mathrm{Str}\,(Q) - \sum_{n=2}^{\infty} (-1)^n \,\mathrm{Str}\left(QP^{n-1}\right) \\
& + \sum_{k=2}^{\infty} \mathrm{Str}\left[Q\left(P^k - \frac{1}{2} P^{k-1}\right) QP^{k-1}\right] + \sum_{k=2}^{\infty} \mathrm{Str}\left[Q\left(P^{2k-1} - P^{2k-2}\right) Q\right] \\
& + \sum_{k=3}^{\infty} \sum_{l=1}^{k-2} \mathrm{Str}\left[Q\left(P^{2k-l-1} - P^{2k-l-2}\right) QP^l\right] \ .
\end{aligned}
\tag{B.41}
$$

Let us show that the quantity $\Re$ is zero. To this end, let us recall the properties (4.1), (4.11), (4.13), (4.14),

$$
\begin{aligned}
&\mathrm{Str}\,(Q_1) \equiv 0\ , \quad \mathrm{Str}\,(R) - \frac{1}{2}\mathrm{Str}\left(Q_1^2\right) \equiv 0\ , \\
&Q = Q_1 + Q_2\ , \quad Q_2 = \mathrm{tr}\,(Y)\ , \quad QP^n = \mathrm{tr}\left[m^{n-1}\,(e+m)\,Y\right]\ , \quad n \geq 1\ ,
\end{aligned}
$$

which imply the relations

$$
\begin{aligned}
&\mathrm{Str}\,(Q) = \mathrm{Str}\,[\mathrm{tr}\,(Y)]\ , \\
&\mathrm{Str}\,(R) - \frac{1}{2}\mathrm{Str}\left(Q^2\right) = -\mathrm{Str}\,[Q_1 \mathrm{tr}\,(Y)] - \frac{1}{2}\mathrm{Str}\,[\mathrm{tr}\,(Y)\,\mathrm{tr}\,(Y)]
\end{aligned}
\tag{B.42}
$$

and

$$
\begin{aligned}
&QPQ = \mathrm{tr}\,[(e+m)\,Y]\,[Q_1 + \mathrm{tr}\,(Y)]\ , \\
&QP = \mathrm{tr}\,[(e+m)\,Y]\ , \quad QP^{n-1} = \mathrm{tr}\left[m^{n-2}\,(e+m)\,Y\right]\ , \\
&Q\left[P^k - (1/2)\,P^{k-1}\right] = \mathrm{tr}\left[m^{k-2}\,(m - e/2)\,(e+m)\,Y\right]\ , \\
&QP^{k-1} = \mathrm{tr}\left[m^{k-2}\,(e+m)\,Y\right]\ , \\
&Q\left(P^{2k-1} - P^{2k-2}\right) = \mathrm{tr}\left[m^{2k-3}\left(m^2 - e\right)Y\right]\ , \\
&Q\left(P^{2k-l-1} - P^{2k-l-2}\right) = \mathrm{tr}\left[\left(m^{2k-l-2} - m^{2k-l-3}\right)(e+m)\,Y\right]\ , \\
&QP^l = \mathrm{tr}\left[m^{l-1}\,(e+m)\,Y\right]\ ,
\end{aligned}
\tag{B.43}
$$

As a consequence, we arrive at the following representation of (B.41):

$$
\Re = \Re_1 + \Re_2 + \Re_3\ , \tag{B.44}
$$

where the contributions $\Re_1$, $\Re_2$,$\Re_3$ are given by

$$
\Re_1 = \mathrm{Str}\,[\mathrm{tr}\,(Y)] - \sum_{n=2}^{\infty} (-1)^n\,\mathrm{Str}\left\{\mathrm{tr}\left[m^{n-2}\,(e+m)\,Y\right]\right\}\ , \tag{B.45}
$$

$$
\Re_2 = -\mathrm{Str}\,[Q_1 \mathrm{tr}\,(Y)] + \mathrm{Str}\,\{Q_1 \mathrm{tr}\,[(e+m)\,Y]\} + \sum_{k=2}^{\infty} \mathrm{Str}\left\{Q_1 \mathrm{tr}\left[m^{2k-3}\left(m^2 - e\right)Y\right]\right\}\ , \tag{B.46}
$$

$$
\begin{aligned}
\Re_3 = &-\frac{1}{2}\mathrm{Str}\,[\mathrm{tr}\,(Y)\,\mathrm{tr}\,(Y)] + \mathrm{Str}\,\{\mathrm{tr}\,[(e+m)\,Y]\,\mathrm{tr}\,(Y)\} \\
&+ \sum_{k=2}^{\infty} \mathrm{Str}\left\{\mathrm{tr}\left[m^{2k-3}\left(m^2 - e\right)Y\right]\mathrm{tr}\,(Y)\right\} \\
&+ \sum_{k=2}^{\infty} \mathrm{Str}\left\{\mathrm{tr}\left[m^{k-2}\,(m - e/2)\,(e+m)\,Y\right]\mathrm{tr}\left[m^{k-2}\,(e+m)\,Y\right]\right\} \\
&+ \sum_{k=3}^{\infty} \sum_{l=1}^{k-2} \mathrm{Str}\left\{\mathrm{tr}\left[\left(m^{2k-l-2} - m^{2k-l-3}\right)(e+m)\,Y\right]\mathrm{tr}\left[m^{l-1}\,(e+m)\,Y\right]\right\}\ .
\end{aligned}
\tag{B.47}
$$

Notice that the operation tr enters $\Re_1$, $\Re_2$ linearly, whereas $\Re_3$ contains the operation tr quadratically. Let us show that $\Re_1$, $\Re_2$, $\Re_3$ are equal to zero.

The contribution $\Re_1$, linear in the elements of the matrix Y, reads equivalently

$$
\Re_1 = \mathrm{Str}\,[\mathrm{tr}\,(Y)] + \mathrm{Str} \sum_{k=1}^{\infty} (-1)^k\,\mathrm{tr}\left[m^{k-1}\,(e+m)\,Y\right] \equiv \mathrm{Str}\,[\mathrm{tr}\,(\mathcal{A}Y)]\ , \tag{B.48}
$$

where

$$\mathcal{A}=e+\sum_{k=1}^{\infty}(-1)^k\,m^{k-1}\,(e+m)=e-\sum_{k=0}^{\infty}(-1)^k\,m^k+\sum_{k=1}^{\infty}(-1)^k\,m^k=e-e\equiv 0\;.$$

Therefore, the contribution $\Re_1$ vanishes identically, $\Re_1\equiv 0$.

The contribution $\Re_2$, bilinear in the elements of the matrices Y and Q_1, reads equivalently

$$\begin{aligned}\Re_2 &= -\mathrm{Str}\,[Q_1\mathrm{tr}\,(Y)]+\mathrm{Str}\,(Q_1\mathrm{tr}\,[(e+m)\,Y])\\ &+\sum_{k=2}^{\infty}\mathrm{Str}\left\{Q_1\mathrm{tr}\left[m^{2k-1}Y\right]\right\}-\sum_{k=2}^{\infty}\mathrm{Str}\left\{Q_1\mathrm{tr}\left[m^{2k-3}Y\right]\right\}\equiv\mathrm{Str}\,[Q_1\mathrm{tr}\,(\mathcal{B}Y)]\;\;,\end{aligned}\tag{B.49}$$

where

$$\mathcal{B}=-e+(e+m)+\sum_{k=2}^{\infty}m^{2k-1}-\sum_{k=2}^{\infty}m^{2k-3}\equiv 0\;,$$

since

$$\sum_{k=2}^{\infty}\left(m^{2k-1}-m^{2k-3}\right)=\sum_{k=2}^{\infty}m^{2k-1}-\sum_{k=1}^{\infty}m^{2k-1}=-m\;.\tag{B.50}$$

Therefore, the contribution $\Re_2$ vanishes identically, $\Re_2\equiv 0$.

The contribution $\Re_3$, quadratic in the elements of the matrix Y, reads equivalently

$$\begin{aligned}\Re_3 &= (1/2)\,\mathrm{Str}\,[\mathrm{tr}\,(Y)\,\mathrm{tr}\,(Y)]+\mathrm{Str}\,[\mathrm{tr}\,(mY)\,\mathrm{tr}\,(Y)]\\ &+\sum_{k=2}^{\infty}\mathrm{Str}\left\{\mathrm{tr}\left[\left(m^{2k-1}-m^{2k-3}\right)Y\right]\mathrm{tr}\,(Y)\right\}\\ &+\sum_{k=2}^{\infty}\mathrm{Str}\left\{\mathrm{tr}\left[\left(m^{k}+m^{k-1}/2-m^{k-2}/2\right)Y\right]\mathrm{tr}\left[\left(m^{k-1}+m^{k-2}\right)Y\right]\right\}\\ &+\sum_{k=3}^{\infty}\sum_{l=1}^{k-2}\mathrm{Str}\left\{\mathrm{tr}\left[\left(m^{2k-l-1}-m^{2k-l-3}\right)Y\right]\mathrm{tr}\left[\left(m^{l-1}+m^{l}\right)Y\right]\right\}\;\;,\end{aligned}\tag{B.51}$$

and therefore the expression for $\Re_3$ has the structure

$$\Re_3=\sum_{n=0}^{\infty}\Re_3^{(n)}\;\;,\quad\text{where}\quad\Re_3^{(n)}=\mathrm{Str}\sum_{k,l}a_{kl}\mathrm{tr}\left(m^kY\right)\mathrm{tr}\left(m^lY\right)\;\;,\quad n=k+l\;.\tag{B.52}$$

Let us examine the following contribution, taking into account the property $\mathrm{Str}\,(AB)=\mathrm{Str}\,(BA)$:

$$\begin{aligned}&\mathrm{Str}\sum_{k=2}^{\infty}\mathrm{tr}\left[\left(m^{k-1}/2-m^{k-2}/2\right)Y\right]\mathrm{tr}\left[\left(m^{k-1}+m^{k-2}\right)Y\right]\\ &=\frac{1}{2}\mathrm{Str}\sum_{k=2}^{\infty}\mathrm{tr}\left(m^{k-1}Y\right)\mathrm{tr}\left(m^{k-1}Y\right)-\frac{1}{2}\mathrm{Str}\sum_{k=2}^{\infty}\mathrm{tr}\left(m^{k-2}Y\right)\mathrm{tr}\left(m^{k-2}Y\right)\\ &=\frac{1}{2}\mathrm{Str}\sum_{k=2}^{\infty}\mathrm{tr}\left(m^{k-1}Y\right)\mathrm{tr}\left(m^{k-1}Y\right)-\frac{1}{2}\mathrm{Str}\sum_{k=1}^{\infty}\mathrm{tr}\left(m^{k-1}Y\right)\mathrm{tr}\left(m^{k-1}Y\right)\\ &=-\frac{1}{2}\mathrm{Str}\left[\mathrm{tr}\left(m^0Y\right)\mathrm{tr}\left(m^0Y\right)\right]=-\frac{1}{2}\mathrm{Str}\,[\mathrm{tr}\,(Y)\,\mathrm{tr}\,(Y)]\;\;.\end{aligned}\tag{B.53}$$

Let us also examine the contribution

$$\mathrm{Str}\sum_{k=2}^{\infty}\mathrm{tr}\left[\left(m^{2k-1}-m^{2k-3}\right)Y\right]\mathrm{tr}\,(Y)=-\mathrm{Str}\,\{[\mathrm{tr}\,(mY)]\,\mathrm{tr}\,(Y)\}\;\;,\tag{B.54}$$

where account has been taken of (B.50). As a consequence of (B.53), (B.54), the terms of $\Re_3$ containing $\mathrm{Str}\left[(\mathrm{tr}\,(Y))^{\,2}\right]$ and $\mathrm{Str}\,[\mathrm{tr}\,(mY)\,\mathrm{tr}\,(Y)]$ are cancelled out, and therefore the expression becomes simplified:

$$\begin{aligned}\Re_3 &= \mathrm{Str}\sum_{k=2}^{\infty}\mathrm{tr}\left[\left(m^{k-1}+m^{k-2}\right)Y\right]\mathrm{tr}\left(m^kY\right)\\ &+\mathrm{Str}\sum_{k=3}^{\infty}\sum_{l=1}^{k-2}\mathrm{tr}\left[\left(m^{2k-l-1}-m^{2k-l-3}\right)Y\right]\mathrm{tr}\left[\left(m^{l-1}+m^l\right)Y\right]\ ,\end{aligned}\tag{B.55}$$

which also means that $\Re_3$ formally starts with the second order in the elements of the matrix m. In more detail, let us examine the constituents of $\Re_3$ in their relation to the order n of expansion in powers of the matrix elements m^a_b:

$k \geq 2:$

$\mathrm{tr}\left(m^{k-2}Y\right)\mathrm{tr}\left(m^kY\right)$	$\mathrm{tr}\left(m^{k-1}Y\right)\mathrm{tr}\left(m^kY\right)$
$n=2k-2$	$n=2k-1$

,

$k \geq 3:$ (B.56)

$\mathrm{tr}\left(m^{2k-l-1}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)$	$\mathrm{tr}\left(m^{2k-l-3}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)$	$\mathrm{tr}\left(m^{2k-l-1}Y\right)\mathrm{tr}\left(m^{l}Y\right)$	$\mathrm{tr}\left(m^{2k-l-3}Y\right)\mathrm{tr}\left(m^{l}Y\right)$
$n=2k-2$	$n=2k-4$	$n=2k-1$	$n=2k-3$

.

For even degrees $n = 2r$:

$$\begin{array}{lllllll}+\mathrm{tr}\left(m^{k-2}Y\right)\mathrm{tr}\left(m^kY\right)\ , & 2k-2=2r\ , & k\geq 2\ , & k=r+1\ , & r\geq 1\ , & \\ +\mathrm{tr}\left(m^{2k-l-1}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)\ , & 2k-2=2r\ , & k\geq 3\ , & k=r+1\ , & r\geq 2\ , & l=1,\ldots,r-1\ ,\\ -\mathrm{tr}\left(m^{2k-l-3}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)\ , & 2k-4=2r\ , & k\geq 3\ , & k=r+2\ , & r\geq 1\ , & l=1,\ldots,r\ .\end{array}\tag{B.57}$$

For odd degrees $n = 2r + 1$:

$$\begin{array}{lllllll}+\mathrm{tr}\left(m^{k-1}Y\right)\mathrm{tr}\left(m^kY\right)\ , & 2k-1=2r+1\ , & k\geq 2\ , & k=r+1\ , & r\geq 1\ , & \\ +\mathrm{tr}\left(m^{2k-l-1}Y\right)\mathrm{tr}\left(m^{l}Y\right)\ , & 2k-1=2r+1\ , & k\geq 3\ , & k=r+1\ , & r\geq 2\ , & l=1,\ldots,r-1\ ,\\ -\mathrm{tr}\left(m^{2k-l-3}Y\right)\mathrm{tr}\left(m^{l}Y\right)\ , & 2k-3=2r+1\ , & k\geq 3\ , & k=r+2\ , & r\geq 1\ , & l=1,\ldots,r\ .\end{array}\tag{B.58}$$

In the case $n = 2$ $(r = 1)$ we have

$$\begin{array}{lll}+\mathrm{tr}\left(m^{k-2}Y\right)\mathrm{tr}\left(m^kY\right)\ , & k=r+1=2\ , & \\ +\mathrm{tr}\left(m^{2k-l-1}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)\ , & k=r+1=2\ , & k\ngeq 3\ ,\\ -\mathrm{tr}\left(m^{2k-l-3}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)\ , & k=r+2=3\ , & l=1\ ,\end{array}\tag{B.59}$$

which implies

$$\Re_3^{(2)}=\mathrm{Str}\left[\mathcal{R}_3^{(2)}\right]\equiv 0\ ,\quad \mathcal{R}_3^{(2)}\equiv\mathrm{tr}\,(Y)\,\mathrm{tr}\left(m^2Y\right)-\mathrm{tr}\left(m^2Y\right)\mathrm{tr}\,(Y)\ ,\tag{B.60}$$

whereas in the case $n = 2r \geq 4$ we have

$$\begin{aligned}&\Re_3^{(2r)}=\mathrm{Str}\left[\mathcal{R}_3^{(2k)}\right]\equiv 0\ ,\\ &\mathcal{R}_3^{(2k)}\equiv\mathrm{tr}\left(m^{(r+1)-2}Y\right)\mathrm{tr}\left(m^{(r+1)}Y\right)+\sum_{l=1}^{r-1}\mathrm{tr}\left(m^{2(r+1)-l-1}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)-\sum_{l=1}^{r}\mathrm{tr}\left(m^{2(r+2)-l-3}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)\\ &\quad+\mathrm{tr}\left(m^{r-2}Y\right)\mathrm{tr}\left(m^{r+1}Y\right)+\sum_{l=1}^{r-1}\mathrm{tr}\left(m^{2r+1-l}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)-\sum_{l=1}^{r}\mathrm{tr}\left(m^{2r+1-l}Y\right)\mathrm{tr}\left(m^{l-1}Y\right)\\ &\quad=\mathrm{tr}\left(m^{r-1}Y\right)\mathrm{tr}\left(m^{r+1}Y\right)-\mathrm{tr}\left(m^{r+1}Y\right)\mathrm{tr}\left(m^{r-1}Y\right)\ ,\quad\text{for}\quad r\geq 2\ .\end{aligned}\tag{B.61}$$

In the case $n = 3$ $(r = 1)$ we have

$$
\begin{aligned}
&+\mathrm{tr}\left(m^{k-1}Y\right)\mathrm{tr}\left(m^{k}Y\right) \ , && k = r+1 = 2 \ , \\
&+\mathrm{tr}\left(m^{2k-l-1}Y\right)\mathrm{tr}\left(m^{l}Y\right) \ , && k = r+1 = 2 \ , \quad k \ngeq 3 \ , \\
&-\mathrm{tr}\left(m^{2k-l-3}Y\right)\mathrm{tr}\left(m^{l}Y\right) \ , && k = r+2 = 3 \ , \quad l = 1 \ ,
\end{aligned} \tag{B.62}
$$

which implies

$$
\Re_3^{(3)} = \mathrm{Str}\left[\mathcal{R}_3^{(3)}\right] \equiv 0 \ , \quad \mathcal{R}_3^{(3)} \equiv \mathrm{tr}\left(mY\right)\mathrm{tr}\left(m^2Y\right) - \mathrm{tr}\left(m^2Y\right)\mathrm{tr}\left(mY\right) \ , \tag{B.63}
$$

whereas in the case $n = 2r+1 \geq 5$ we have

$$
\begin{aligned}
\Re_3^{(2r+1)} &= \mathrm{Str}\left[\mathcal{R}_3^{(2r+1)}\right] \equiv 0 \ , \\
\mathcal{R}_3^{(2k)} &\equiv \mathrm{tr}\left(m^{(r+1)-1}Y\right)\mathrm{tr}\left(m^{r+1}Y\right) + \sum_{l=1}^{r-1}\mathrm{tr}\left(m^{2(r+1)-l-1}Y\right)\mathrm{tr}\left(m^{l}Y\right) - \sum_{l=1}^{r}\mathrm{tr}\left(m^{2(r+2)-l-3}Y\right)\mathrm{tr}\left(m^{l}Y\right) \\
&\quad + \mathrm{tr}\left(m^{r}Y\right)\mathrm{tr}\left(m^{r+1}Y\right) + \sum_{l=1}^{r-1}\mathrm{tr}\left(m^{2r+1-l}Y\right)\mathrm{tr}\left(m^{l}Y\right) - \sum_{l=1}^{r}\mathrm{tr}\left(m^{2r+1-l}Y\right)\mathrm{tr}\left(m^{l}Y\right) \\
&\quad = \mathrm{tr}\left(m^{r}Y\right)\mathrm{tr}\left(m^{r+1}Y\right) - \mathrm{tr}\left(m^{r+1}Y\right)\mathrm{tr}\left(m^{r}Y\right) \ , \quad \text{for} \quad r \geq 2 \ .
\end{aligned} \tag{B.64}
$$

Collecting the above results, we can state that

$$
\Re_3^{(2r)} = \Re_3^{(2r+1)} \equiv 0 \ , \quad r = 0, 1, 2 \ldots \ , \tag{B.65}
$$

which implies that the contribution $\Re_3$ is an identical zero:

$$
\Re_3 = \sum_{n=0}^{\infty} \Re_3^{(n)} \equiv 0 \ . \tag{B.66}
$$

B.7 Proof of Lemma 6

Let us suppose $s^a\lambda_a = -s^2\Lambda$ with anticommuting s^a and a certain even-valued Λ. Using the consequent nilpotency $s^{a_1}\cdots s^{a_n} \equiv 0$, $n \geq 3$, the obvious property $s^2\lambda_a \equiv 0$, and the general relation

$$
s^a\left(AB\right) = \left(s^aA\right)B\left(-1\right)^{\varepsilon_B} + A\left(s^aB\right) \ ,
$$

we can write down identically

$$
s^{a_1}s^{a_2}\left(\lambda_{a_1}\lambda_{a_2}\right) = \left(s^{a_1}\lambda_{a_1}\right)\left(s^{a_2}\lambda_{a_2}\right) - \left(s^{a_2}\lambda_{a_1}\right)\left(s^{a_1}\lambda_{a_2}\right) = \left(s^a\lambda_a\right)^2 - \mathrm{tr}\left(m^2\right) \ , \tag{B.67}
$$

$$
\begin{aligned}
s^{a_1}s^{a_2}s^{a_3}\left(\lambda_{a_1}\lambda_{a_2}\lambda_{a_3}\right) &= \left(s^{a_1}\lambda_{a_1}\right)\left(s^{a_2}\lambda_{a_2}\right)\left(s^{a_3}\lambda_{a_3}\right) + \left(s^{a_3}\lambda_{a_1}\right)\left(s^{a_1}\lambda_{a_2}\right)\left(s^{a_2}\lambda_{a_3}\right) \\
&\quad + \left(s^{a_3}\lambda_{a_2}\right)\left(s^{a_2}\lambda_{a_1}\right)\left(s^{a_1}\lambda_{a_3}\right) - \left(s^{a_2}\lambda_{a_2}\right)\left(s^{a_3}\lambda_{a_1}\right)\left(s^{a_1}\lambda_{a_3}\right) \\
&\quad - \left(s^{a_1}\lambda_{a_1}\right)\left(s^{a_3}\lambda_{a_2}\right)\left(s^{a_2}\lambda_{a_3}\right) - \left(s^{a_3}\lambda_{a_3}\right)\left(s^{a_2}\lambda_{a_1}\right)\left(s^{a_1}\lambda_{a_2}\right) \\
&= \left(s^a\lambda_a\right)^3 + 2\mathrm{tr}\left(m^3\right) - 3\left(s^a\lambda_a\right)\mathrm{tr}\left(m^2\right) \equiv 0 \ ,
\end{aligned} \tag{B.68}
$$

$$
\ldots
$$

$$
s^{a_1}\cdots s^{a_n}\left(\lambda_{a_1}\cdots\lambda_{a_n}\right) = \left(s^a\lambda_a\right)^n + K_{n|0}\mathrm{tr}\left(m^n\right) + \sum_{k=1}^{n-2} K_{n|k}\left(s^a\lambda_a\right)^k\mathrm{tr}\left(m^{n-k}\right) \equiv 0 \ , \quad n \geq 3 \ , \tag{B.69}
$$

where $K_{n|k}$, $k = 1, \ldots, n-1$, are certain combinatorial coefficients, whose specific form is not essential here, and $K_{n|0}$ is given by

$$
K_{n|0} = (n-1)!\left(-1\right)^{n-1} \ . \tag{B.70}
$$

Indeed, the term $\left(s^{a}\lambda_{a}\right)^{n}$ is generated as follows:

$$
\begin{aligned}
s^{a_1}\cdots s^{a_n}\left(\lambda_{a_1}\cdots\lambda_{a_n}\right) &= s^{a_2}\cdots s^{a_n}s^{a_1}\left(\lambda_{a_2}\cdots\lambda_{a_n}\lambda_{a_1}\right)=\left(s^{a_2}\cdots s^{a_n}\lambda_{a_2}\cdots\lambda_{a_n}\right)\left(s^{a_1}\lambda_{a_1}\right)+\cdots\\
&=\left(s^{a_1}\lambda_{a_1}\right)s^{a_3}\cdots s^{a_n}s^{a_2}\left(\lambda_{a_3}\cdots\lambda_{a_n}\lambda_{a_2}\right)+\cdots\\
&=\left(s^{a_1}\lambda_{a_1}\right)\left(s^{a_2}\lambda_{a_2}\right)s^{a_3}\cdots s^{a_n}\left(\lambda_{a_3}\cdots\lambda_{a_n}\right)+\cdots\\
&\cdots\\
&=\left(s^{a_1}\lambda_{a_1}\right)\left(s^{a_2}\lambda_{a_2}\right)\cdots\left(s^{a_n}\lambda_{a_n}\right)+\cdots=\left(s^{a}\lambda_{a}\right)^{n}+\cdots\ ,
\end{aligned}
\tag{B.71}
$$

whereas the term $K_{n|0}\mathrm{tr}\left(m^{n}\right)$ can be traced back to

$$
\begin{aligned}
s^{a_1}\cdots s^{a_n}\left(\lambda_{a_1}\cdots\lambda_{a_n}\right) &= s^{a_2}\cdots s^{a_n}s^{a_1}\left(\lambda_{a_1}\cdots\lambda_{a_n}\right)(-1)^{n-1}=\left(s^{a_1}\lambda_{a_n}\right)s^{a_2}\cdots s^{a_n}\left(\lambda_{a_1}\cdots\lambda_{a_{n-1}}\right)(-1)^{n-1}+\cdots\\
&=\left(s^{a_1}\lambda_{a_n}\right)\left(s^{a_n}\lambda_{a_{n-1}}\right)s^{a_2}\cdots s^{a_{n-1}}\left(\lambda_{a_1}\cdots\lambda_{a_{n-2}}\right)(-1)^{n-1}+\cdots\\
&=\left(s^{a_1}\lambda_{a_n}\right)\left(s^{a_n}\lambda_{a_{n-1}}\right)\left(s^{a_{n-1}}\lambda_{a_{n-2}}\right)\cdots\left(s^{a_2}\lambda_{a_1}\right)(-1)^{n-1}+\cdots\ ,
\end{aligned}
\tag{B.72}
$$

and therefore, collecting equal contributions with the leading terms as above, we arrive at

$$
\begin{aligned}
&K_{n|0}\mathrm{tr}\left(m^{n}\right)=\sum_{P}P\left(A_{n\cdots 2}\right)=(n-1)!A_{n\cdots 2}\ ,\\
&A_{n\cdots k\cdots 2}=\left(s^{a_1}\lambda_{a_n}\right)\left(s^{a_n}\lambda_{a_{n-1}}\right)\left(s^{a_{n-1}}\lambda_{a_{n-2}}\right)\cdots\left(s^{a_{k+1}}\lambda_{a_k}\right)\cdots\left(s^{a_3}\lambda_{a_2}\right)\left(s^{a_2}\lambda_{a_1}\right)\ ,
\end{aligned}
\tag{B.73}
$$

where $P\left(A_{n\cdots 2}\right)$ is an arbitrary permutation of the indices in $A_{n\cdots 2}$ corresponding to $a_n,\ldots,a_2$ in (B.73).

From (B.67), (B.68), it follows that the contributions $\mathrm{tr}(m^2)$, $\mathrm{tr}(m^3)$ are BRST-antiBRST-exact:

$$
\mathrm{tr}\left(m^{2}\right)=\left(s^{2}\Lambda\right)^{2}-\frac{1}{2}s^{2}\left(\lambda^{2}\right)\equiv -s^{2}\Lambda_{2}\ , \tag{B.74}
$$

$$
\mathrm{tr}\left(m^{3}\right)=\frac{1}{2}\left(s^{2}\Lambda\right)^{3}-\frac{3}{2}\left(s^{2}\Lambda\right)\mathrm{tr}\left(m^{2}\right)=-\left(s^{2}\Lambda\right)^{3}+\frac{3}{4}\left(s^{2}\Lambda\right)s^{2}\left(\lambda^{2}\right)\equiv -s^{2}\Lambda_{3}\ . \tag{B.75}
$$

Proceeding by induction in the general case $n\geq 2$ and assuming $\mathrm{tr}(m^k)$, $k=1,\ldots,n$, to be BRST-antiBRST-exact, $\mathrm{tr}(m^k)=-s^2\Lambda_k$, we can now prove, by using the relation (B.69) and the identity $s^{a_1}\cdots s^{a_{n+1}}\left(\lambda_{a_1}\cdots\lambda_{a_{n+1}}\right)\equiv 0$, the fact that

$$
\mathrm{tr}\left(m^{n+1}\right)=(-1)^{n}K^{-1}_{n+1|0}\left(s^{2}\Lambda\right)^{n+1}+\sum_{k=1}^{n-1}(-1)^{k}K^{-1}_{n+1|0}K_{n+1|k}\left(s^{2}\Lambda\right)^{k}\left(s^{2}\Lambda_{n+1-k}\right)\equiv -s^{2}\Lambda_{n+1}\ , \tag{B.76}
$$

whence the contribution $\mathrm{tr}(m^{n+1})$ is also BRST-antiBRST-exact, which proves Lemma 6.

B.8 Proof of Lemma 7

Let us consider an odd-valued doublet ψ_a subject to the condition $s^a\psi_a=0$. Making in (B.74)–(B.76) the substitution $\lambda_a=\psi_a$, $\Lambda=0$, $m\equiv m_\psi$, we obtain

$$
\begin{aligned}
&\mathrm{tr}\left(m_{\psi}^{2}\right)=-\frac{1}{2}s^{2}\left(\psi^{2}\right)\ ,\\
&\mathrm{tr}\left(m_{\psi}^{3}\right)=0\ ,\\
&\cdots\\
&\mathrm{tr}\left(m_{\psi}^{n}\right)=0\ ,\quad n\geq 3\ ,
\end{aligned}
\tag{B.77}
$$

which proves the relations (4.27) of Lemma 7. This allows one to make an explicit calculation of the corresponding quantity $\Im$, parameterized by the functional parameters (Λ,ψ_a). Indeed, due to the relations

$$\mathrm{tr}\left(m_{\Lambda}+m_{\psi}\right)^{n}=\mathrm{tr}\sum_{k=0}^{n}C_{n}^{k}f^{n-k}m_{\psi}^{k}\ ,\quad \left(m_{\Lambda}\right)_{b}^{a}=s^{a}s_{b}\Lambda=\delta_{b}^{a}f\ ,\quad \left(m_{\psi}\right)_{b}^{a}=s^{a}\psi_{b}\ , \tag{B.78}$$

the corresponding quantity $\Im=\Im\left(\Lambda,\psi\right)$ reads

$$\Im\left(\Lambda,\psi\right)=\ln\left(1-\frac{1}{2}s^{2}\Lambda\right)^{-2}+M\left(\Lambda,\psi\right)\ ,\quad M\left(\Lambda,\psi\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}}{n}\sum_{k=1}^{n}C_{n}^{k}f^{n-k}\mathrm{tr}\left[\left(m_{\psi}\right)^{k}\right]_{f=-\frac{1}{2}s^{2}\Lambda}\ . \tag{B.79}$$

The only nontrivial quantity $\mathrm{tr}(m_{\psi}^{2})\not\equiv 0$ amongst $\mathrm{tr}(m_{\psi}^{n})$ leads to

$$\begin{aligned}M\left(\Lambda,\psi\right)&=-\mathrm{tr}\left[\left(m_{\psi}\right)\right]+\mathrm{tr}\left(m_{\psi}^{2}\right)\sum_{n=2}^{\infty}\frac{\left(-1\right)^{n}}{n}\ C_{n}^{2}f^{n-2}\big|_{f=-\frac{1}{2}s^{2}\Lambda}\\&=\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)+\mathrm{tr}\left(m_{\psi}^{2}\right)\sum_{n=3}^{\infty}\frac{\left(-1\right)^{n}}{n}\ C_{n}^{2}f^{n-2}\big|_{f=-\frac{1}{2}s^{2}\Lambda}=\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)+\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)\sum_{k=1}^{\infty}\left(-1\right)^{k}\left(k+1\right)\,f^{k}\big|_{f=-\frac{1}{2}s^{2}\Lambda}\ ,\end{aligned} \tag{B.80}$$

where

$$\sum_{k=1}^{\infty}\left(-1\right)^{k}\left(k+1\right)x^{k}=-\sum_{k=2}^{\infty}\left(-1\right)^{k}kx^{k-1}=-\frac{\partial}{\partial x}\sum_{k=2}^{\infty}\left(-1\right)^{k}x^{k}=-\frac{\partial}{\partial x}\left[\left(1+x\right)^{-1}-1+x\right]=\left(1+x\right)^{-2}-1\ . \tag{B.81}$$

Therefore,

$$\begin{aligned}M\left(\Lambda,\psi\right)&=\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)+\mathrm{tr}\left(m_{\psi}^{2}\right)\sum_{n=3}^{\infty}\frac{\left(-1\right)^{n}}{n}\ C_{n}^{2}f^{n-2}\big|_{f=-\frac{1}{2}s^{2}\Lambda}=\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)+\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)\sum_{k=1}^{\infty}\left(-1\right)^{k}\left(k+1\right)\,f^{k}\big|_{f=-\frac{1}{2}s^{2}\Lambda}\\&=\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)\left[1+\sum_{k=1}^{\infty}\left(-1\right)^{k}\left(k+1\right)f^{k}\right]_{f=-\frac{1}{2}s^{2}\Lambda}=\frac{1}{2}\mathrm{tr}\left(m_{\psi}^{2}\right)\left[\left(1-\frac{1}{2}s^{2}\Lambda\right)^{-2}\right]\ .\end{aligned} \tag{B.82}$$

This result describes the contribution to $\Im\left(\Lambda,\psi\right)$ caused by the arbitrariness in the solutions of $s^{a}\left(\lambda_{a}-s_{a}\Lambda\right)=0$, with a given Λ. This contribution is BRST-antiBRST-exact due to the fact that $\mathrm{tr}(m_{\psi}^{2})=-\left(1/2\right)s^{2}\left(\psi^{2}\right)$:

$$M\left(\Lambda,\psi\right)=s^{2}N\left(\Lambda,\psi\right)\ . \tag{B.83}$$

The relations (B.79), (B.82) prove (4.28), which finishes the proof of Lemma 7.

B.9 Proof of Lemma 8

Let us examine the equation (4.29),

$$\ln\left(1-\frac{1}{2}s^{2}\Lambda\right)^{-2}-\frac{1}{4}s^{2}\left(\psi^{2}\right)\left(1-\frac{1}{2}s^{2}\Lambda\right)^{-2}=\frac{1}{2i\hbar}s^{2}\Delta F\ ,$$

for an unknown functional $\Lambda=\Lambda\left(\Delta F,\psi\right)$ and introduce the following notation:

$$\begin{aligned}&\Lambda_{0}:\ln\left(1-\frac{1}{2}s^{2}\Lambda_{0}\right)^{-2}=\frac{1}{2i\hbar}s^{2}\Delta F\ ,\\&\frac{1}{4}s^{2}\left(\psi^{2}\right)\equiv\gamma\ ,\quad X\equiv\left(1-\frac{1}{2}s^{2}\Lambda\right)^{-2}=X_{0}+\Delta X\ ,\\&X\equiv\left(1-\frac{1}{2}s^{2}\Lambda\right)^{-2}\Longrightarrow\ln X_{0}=\frac{1}{2i\hbar}s^{2}\Delta F\ ,\end{aligned} \tag{B.84}$$

whence follows a chain of relations:

$$
\begin{aligned}
&\ln X - \gamma X = \frac{1}{2i\hbar} s^2 \Delta F \ , \\
&\ln\left(X_0 + \Delta X\right) - \gamma\left(X_0 + \Delta X\right) = \frac{1}{2i\hbar} s^2 \Delta F \ , \\
&\ln X_0 + \ln\left(1 + \frac{\Delta X}{X_0}\right) - \gamma X_0 \left(1 + \frac{\Delta X}{X_0}\right) = \frac{1}{2i\hbar} s^2 \Delta F \ , \\
&\ln\left(1 + \frac{\Delta X}{X_0}\right) = \gamma X_0 \left(1 + \frac{\Delta X}{X_0}\right) \ .
\end{aligned} \tag{B.85}
$$

Let us introduce a new function,

$$
\theta\left(x\right) = \frac{\ln\left(1+x\right)}{\left(1+x\right)} \ , \quad \theta\left(0\right) = 0 \ , \tag{B.86}
$$

and the inverse function:

$$
\vartheta\left(y\right) : \vartheta\left(\theta\left(x\right)\right) = x \ , \quad \vartheta\left(0\right) = 0 \ . \tag{B.87}
$$

Hence,

$$
\theta\left(\Delta X/X_0\right) = \gamma X_0 \Longrightarrow \Delta X/X_0 = \vartheta\left(\gamma X_0\right) \Longrightarrow \Delta X = X_0 \cdot \vartheta\left(\gamma X_0\right) \ , \tag{B.88}
$$

which implies

$$
X = X_0 + \Delta X = X_0 + X_0 \cdot \vartheta\left(\gamma X_0\right) = X_0\left[1 + \vartheta\left(\gamma X_0\right)\right] \tag{B.89}
$$

and ensures

$$
X\left(X_0, \gamma\right) = X_0\left[1 + \vartheta\left(\gamma X_0\right)\right] \Longrightarrow X\left(X_0, 0\right) = X_0 \ . \tag{B.90}
$$

This implies

$$
\ln X = \ln X_0 + \ln\left[1 + \vartheta\left(\gamma X_0\right)\right] = \frac{1}{2i\hbar} s^2 \Delta F + \ln\left[1 + \vartheta\left(\gamma X_0\right)\right] \tag{B.91}
$$

Recalling (B.84),

$$
X = \left(1 - \frac{1}{2} s^2 \Lambda\right)^{-2} \ ,
$$

we arrive at a chain of relations:

$$
\begin{aligned}
&\ln\left(1 - \frac{1}{2} s^2 \Lambda\right) = -\frac{1}{4i\hbar} s^2 \Delta F - \frac{1}{2} \ln\left[1 + \vartheta\left(\gamma X_0\right)\right] \ , \\
&1 - \frac{1}{2} s^2 \Lambda = \exp\left(-\frac{1}{4i\hbar} s^2 \Delta F\right) \exp\left\{\ln\left[1 + \vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}}\right\} \ , \\
&1 - \frac{1}{2} s^2 \Lambda = \exp\left(-\frac{1}{4i\hbar} s^2 \Delta F\right) \left[1 + \vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}} \ ,
\end{aligned} \tag{B.92}
$$

whence

$$
s^2 \Lambda\left(\Delta F, \gamma\right) = 2\left\{1 - \left[1 + \vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}} \exp\left[\left(i/4\hbar\right) s^2 \Delta F\right]\right\} \ . \tag{B.93}
$$

In the case $s^2 \Delta F \neq 0$, a solution Λ to this equation can be found as

$$
\Lambda\left(\Delta F, \psi\right) = \frac{2\Delta F}{s^2 \Delta F} \left\{1 - \left[1 + \vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}} \exp\left[\left(i/4\hbar\right) s^2 \Delta F\right]\right\} \ , \tag{B.94}
$$

where it must be recalled that (B.84)

$$
X_0 = \exp\left(\frac{1}{2i\hbar} s^2 \Delta F\right) \ , \quad \gamma = \frac{1}{4} s^2 \left(\psi^2\right) \ .
$$

Let us now examine the case $s^2\Delta F = 0$:

$$s^2\Lambda = 2\left\{1 - \left[1+\vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}} \exp\left[\left(i/4\hbar\right)s^2\Delta F\right]\right\}\Big|_{s^2\Delta F=0}$$
$$= 2\left\{1 - \left[1+\vartheta\left(\gamma\right)\right]^{-\frac{1}{2}}\right\} \ , \tag{B.95}$$

whence there are two possibilities:

$$\gamma = 0: \quad s^2\Lambda = 0 \Longrightarrow \Lambda = s^a\tilde{\lambda}_a + s^2\tilde{\Lambda} \ , \tag{B.96}$$
$$\gamma \neq 0: \quad s^2\Lambda = 2\left\{1 - \left[1+\vartheta\left(\gamma\right)\right]^{-\frac{1}{2}}\right\} \ , \tag{B.97}$$

which, in the latter case, implies

$$\Lambda\left(\psi\right) = \frac{2\psi^2}{s^2\left(\psi^2\right)}\left\{1 - \left[1+\vartheta\left(\gamma\right)\right]^{-\frac{1}{2}}\right\}_{\gamma=\frac{1}{4}s^2\left(\psi^2\right)} \ . \tag{B.98}$$

Summarizing the relations (B.94), (B.96), (B.97), (B.98) and the respective cases $\Lambda = 0$, $s^2\left(\psi^2\right) = 0$ of (4.29), (B.94), we have

$$\begin{array}{ll} \text{a) } s^2\Delta F \neq 0: & \Lambda\left(\Delta F,\psi\right) = \frac{2\Delta F}{s^2\Delta F}\left\{1 - \left[1+\vartheta\left(\gamma X_0\right)\right]^{-\frac{1}{2}} X_0^{-\frac{1}{2}}\right\}\Big|_{X_0=\exp\left(\frac{1}{2i\hbar}s^2\Delta F\right),\ \gamma=\frac{1}{4}s^2\left(\psi^2\right)} \\ \text{b) } s^2\Delta F = 0\ ,\ s^2\left(\psi^2\right) = 0: & \Lambda = s^a\tilde{\lambda}_a + s^2\tilde{\Lambda}\ , \\ \text{c) } s^2\Delta F = 0,\ s^2\left(\psi^2\right) \neq 0: & \Lambda\left(\psi\right) = \frac{2\psi^2}{s^2\left(\psi^2\right)}\left\{1 - \left[1+\vartheta\left(\gamma\right)\right]^{-\frac{1}{2}}\right\}\Big|_{\gamma=\frac{1}{4}s^2\left(\psi^2\right)}\ , \\ \text{d) } \Lambda\left(\Delta F,\psi\right) = 0: & -\frac{1}{4}s^2\left(\psi^2\right) = \frac{1}{2i\hbar}s^2\Delta F\ , \\ \text{e) } s^2\left(\psi^2\right) = 0,\ s^2\Delta F \neq 0: & \Lambda\left(\Delta F, 0\right) = \frac{2\Delta F}{s^2\Delta F}\left[1 - \exp\left(\frac{i}{4\hbar}s^2\Delta F\right)\right]\ , \end{array} \tag{B.99}$$

where the relations a), b), c) in (B.99) thereby prove Lemma 8.

B.10 Proof of Lemma 9

Using the property $\mathrm{Str}\left(AB\right) = \mathrm{Str}\left(BA\right)$ for even matrices, we examine the quantities $\mathrm{Str}\left(\mathcal{U}^{n-1}\mathcal{W}\right) = \mathrm{Str}\left(\mathcal{W}\mathcal{U}^{n-1}\right)$, $n > 1$, where

$$\mathcal{U}^{\mathsf{p}}_{\mathsf{q}} = \mathcal{X}^{\mathsf{p}a}\lambda_{a,q} = \left(\mathsf{s}^a\Gamma^{\mathsf{p}}\right)\lambda_{a,q}\ , \quad \mathcal{W}^{\mathsf{p}}_{\mathsf{q}} = -\frac{1}{2}\lambda^2\mathcal{Y}^{\mathsf{p}}_{,\mathsf{q}} = \frac{1}{4}\lambda^2\left(\mathsf{s}^2\Gamma^{\mathsf{p}}\right)_{,\mathsf{q}}\ ,$$

and write down a chain of relations, taking account of $\lambda_{b,p}\mathcal{X}^{\mathsf{p}a} = \mathsf{s}^a\lambda_b = \mathsf{m}^a_b$:

$$\begin{aligned} n = 2:&\ \left(\mathcal{W}\mathcal{U}\right)^p_q = \mathcal{W}^p_r\mathcal{U}^r_q = \frac{1}{4}\lambda^2\left(\mathsf{s}^2\Gamma^{\mathsf{p}}\right)_{,\mathsf{r}}\mathcal{X}^{\mathsf{r}a}\lambda_{a,q} = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\lambda_{a,q} \\ n = 3:&\ \left(\mathcal{W}\mathcal{U}^2\right)^p_q = \left(\mathcal{W}\mathcal{U}\right)^p_r\mathcal{U}^r_q = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\left(\lambda_{a,r}\mathcal{X}^{rb}\right)\lambda_{b,q} = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\mathsf{m}^b_a\lambda_{b,q}\ , \\ n = 4:&\ \left(\mathcal{W}\mathcal{U}^3\right)^p_q = \left(\mathcal{W}\mathcal{U}^2\right)^p_r\mathcal{U}^r_q = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\mathsf{m}^b_a\left(\lambda_{b,r}\mathcal{X}^{rc}\right)\lambda_{c,q} = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\left(\mathsf{m}^2\right)^b_a\lambda_{b,q} \\ &\ \ldots \\ n \geq 2:&\ \left(\mathcal{W}\mathcal{U}^{n-1}\right)^p_q = \left(\mathcal{W}\mathcal{U}^2\right)^p_r\mathcal{U}^r_q = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\left(\mathsf{m}^{n-2}\right)^b_a\lambda_{b,q}\ , \end{aligned} \tag{B.100}$$

whence

$$\mathrm{Str}\left(\mathcal{U}^{n-1}\mathcal{W}\right) = \left(\mathcal{W}\mathcal{U}^{n-1}\right)^p_p(-1)^{\varepsilon_p} = \frac{1}{4}\lambda^2\left(\mathsf{s}^a\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\left(\mathsf{m}^{n-2}\right)^b_a\lambda_{b,p}(-1)^{\varepsilon_p} = -\frac{1}{4}\lambda_{a,p}\left(\mathsf{m}^{n-2}\right)^a_b\left(\mathsf{s}^b\mathsf{s}^2\Gamma^{\mathsf{p}}\right)\lambda^2\ , \quad n > 1\ ,$$

which thereby proves Lemma 9.

B.11 Proof of Lemma 10

Let us establish the relation (4.69) between the matrices $\mathcal{V}_1$ and $\mathcal{W}$ in (4.1). To do so, we use the generating equations (2.55) and represent the condition of invariance of the integrand $\mathcal{I}_{\Gamma}^{(F)}$ in (2.53) under the BRST-antiBRST transformations $\delta\Gamma^p = (\mathsf{s}^a\Gamma^p)\,\mu_a = \mathcal{X}^{pa}\mu_a$ in the form, being a reformulation of (2.64),

$$\mathcal{S}_{F,p}\mathcal{X}^{\mathsf{p}a} = i\hbar\mathcal{X}^{\mathsf{p}a}_{,\mathsf{p}}\ , \quad \text{where} \quad \mathcal{X}^{\mathsf{p}a}_{,\mathsf{p}} = -\Delta^a S\ . \tag{B.101}$$

Let us write down identically:

$$\begin{aligned}\mathrm{Str}\left(\mathcal{V}_1\right) + \mathrm{Str}\left(\mathcal{W}\right) - \frac{1}{2}\mathrm{Str}\left(\mathcal{V}_1^2\right) &= \left[\left(\mathcal{V}_1\right)^{\mathsf{p}}_{\mathsf{p}} + \mathcal{W}^{\mathsf{p}}_{\mathsf{p}} - \frac{1}{2}\left(\mathcal{V}_1\right)^{\mathsf{p}}_{\mathsf{q}}\left(\mathcal{V}_1\right)^{\mathsf{q}}_{\mathsf{p}}\right](-1)^{\varepsilon_{\mathsf{p}}} \\ &= \mathcal{X}^{\mathsf{p}a}_{,\mathsf{p}}\lambda_a - \frac{1}{2}(-1)^{\varepsilon_{\mathsf{p}}}\left(\mathcal{Y}^{\mathsf{p}}_{,\mathsf{p}} - \frac{1}{2}\mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}_{,\mathsf{p}}\varepsilon_{ba}\right)\lambda^2\ .\end{aligned} \tag{B.102}$$

Considering

$$\begin{aligned}\mathcal{Y}^{\mathsf{p}}_{,\mathsf{p}} - \frac{1}{2}\mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}_{,\mathsf{p}}\varepsilon_{ba} &= \frac{1}{2}\varepsilon_{ba}\left(\mathcal{X}^{\mathsf{p}a}_{,\mathsf{qp}}\mathcal{X}^{\mathsf{q}b}(-1)^{\varepsilon_{\mathsf{p}}(\varepsilon_{\mathsf{q}}+1)} + \mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}_{,\mathsf{p}}\right) - \frac{1}{2}\varepsilon_{ba}\mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}_{,\mathsf{p}} \\ &= \frac{1}{2}\varepsilon_{ba}\left(\mathcal{X}^{\mathsf{p}a}_{,\mathsf{qp}}\mathcal{X}^{\mathsf{q}b}(-1)^{\varepsilon_{\mathsf{p}}(\varepsilon_{\mathsf{q}}+1)} + \mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}_{,\mathsf{p}} - \mathcal{X}^{\mathsf{p}a}_{,\mathsf{q}}\mathcal{X}^{\mathsf{q}b}_{,\mathsf{p}}\right) = \frac{1}{2}\varepsilon_{ba}\mathcal{X}^{\mathsf{p}a}_{,\mathsf{pq}}\mathcal{X}^{\mathsf{q}b}(-1)^{\varepsilon_{\mathsf{p}}}\ ,\end{aligned} \tag{B.103}$$

we arrive at

$$\mathrm{Str}\left(\mathcal{V}_1\right) + \mathrm{Str}\left(\mathcal{W}\right) - \frac{1}{2}\mathrm{Str}\left(\mathcal{V}_1^2\right) = \mathcal{X}^{\mathsf{p}a}_{,\mathsf{p}}\lambda_a + \frac{1}{4}\varepsilon_{ab}\mathcal{X}^{\mathsf{p}a}_{,\mathsf{pq}}\mathcal{X}^{\mathsf{q}b}\lambda^2\ , \tag{B.104}$$

where (B.101) implies

$$\mathcal{X}^{\mathsf{p}a}_{,\mathsf{p}} = -\Delta^a S\ , \quad \mathcal{X}^{\mathsf{p}a}_{,\mathsf{pq}}\mathcal{X}^{\mathsf{q}b} = -\left(\Delta^a S\right)_{,\mathsf{p}}\mathcal{X}^{\mathsf{p}b} = -\mathsf{s}^b\left(\Delta^a S\right)\ , \quad \text{where} \ \ G_{,\mathsf{p}}\mathcal{X}^{\mathsf{p}a} = G_{,\mathsf{p}}\left(\mathsf{s}^a\Gamma^{\mathsf{p}}\right) = \mathsf{s}^a G\ . \tag{B.105}$$

Hence,

$$\mathrm{Str}\left(\mathcal{V}_1\right) + \mathrm{Str}\left(\mathcal{W}\right) - \frac{1}{2}\mathrm{Str}\left(\mathcal{V}_1^2\right) = -\left(\Delta^a S\right)\lambda_a - \frac{1}{4}\left(\mathsf{s}_a\Delta^a S\right)\lambda^2\ ,$$

which thereby proves Lemma 10.

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